MATH 0235
From Professor Lennard

I will give out a more detailed Course Description on the first day of class.

Also, any student interested in joining 0235 is very welcome to e-mail me directly at lennard@pitt.edu.

Due to the coronavirus, this course will almost surely have on-line aspects in Fall 2020. I will announce them later.

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To students interested in Calculus at Pitt:

There are many Fall Calculus options at Pitt:

e.g. [1] regular Calc 1 (Math 0220);

[2] going straight into regular Calc 2 (Math 0230), if you have high enough AP high school calculus scores; or

[3] Honors Math 0235

I believe this is the most challenging option…; and hopefully it is a very interesting option… !

My philosophy and belief is that math is fun, creative and beautiful - both a science and an art.

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Math 0235.

This Honors course covers all of 1-variable Calculus, and leads to (multi-variable) Calculus 3, and the beginning theoretical math analysis courses: Math 0413/0420 or Honors Math 0450.

M0235 is designed for highly motivated students, and replaces M0220 and M0230 (regular Calc 1 and 2). Moreover we cover the same material - and more (and in the same depth - and more). There is too much material to cover it all, in detail, in lectures... A lot of the course homework will consist of assigned reading, noting and learning from our textbook - and/or elsewhere - of material not covered in class.

Lecture notes: I recommend that you take clear and careful lecture notes; they will be the core of the class; and your main reference when studying for the in-term and final exams.

I expect a lot from you... You are not competing against each other - since I do not "curve.'' You will be simply striving to learn the material to the best of your ability, knowing the absolute grade levels in advance.

A previous serious encounter with Calculus - where you learned a lot – will be a great help in the class... A strong knowledge of pre-calculus algebra, trigonometry and plane geometry plays an important role in doing well in many math courses, and this course is no exception. I will assume that you have this knowledge. If this is not the case, then the first week of term is the best time to begin your revision. For example, I expect you to know all your trig formulas...

You will be required also to use and apply logic and algebra to prove certain mathematical statements and formulas in this course. Be aware of this from the very beginning.

I have attached a sample of my 0235 Fall 2008 Lecture notes (lectures 12 through 21).

My lecture notes may vary from year to year... Nevertheless, this will give you an idea of the type of material you will be expected to learn...

Our Fall 2020 text is: “Calculus, 6th edition, Single and Multivariable", by D. Hughes-Hallett, A.M. Gleason, W.G. McCallum, et al; John Wiley & Sons, Inc.. Or, buy the 5th Edition, which is essentially the same… Both will work.

I will cover much of the material in Chapters 1 through 10 of Hughes-Hallet et al – and possibly parts of some other sections; but not necessarily in the same manner or order. Also, the course will include: any extra topics or modifications to the material or approach in the text that I make in class.

Note that regular calculus at Pitt uses a different textbook, written by Stewart...]

PROBLEMS AND HOMEWORK:

Homework Assignments are not for credit. They are given to help you prepare yourself for the exams.

I will regularly assign problems. You should regularly solve these assigned problems...

Please consult me or your Teaching Assistant regularly - in office hours or recitation - about problems you are "stuck on''; or when you are uncertain of your solutions. Also, you can go to the Engineering/Science/Math Library in Benedum Hall and consult the many books there about Calculus... Or, try "Google" to find mathematics resources: books, research papers, on-line encyclopedias, etc...

Do not be surprised if you are spending many more hours per week on Math 0235 homework than you did on any previous math homework...

I encourage students to form study groups, to solve problems together and learn from each other.

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Best wishes,

Dr. Lennard.

FALL '08 : 0235 HONORS CALCULUS : LECTURE NOTES : PART 2 : 14/OCT

CHRIS LENNARD

12. Math 0235 : Lecture 12 : Mon 22/September/08

"Power series" and "The Super-Fact"...

Henceforth, we will often abbreviate the word "respectively" to *"resp."***.**

Fix an arbitrary $c \in \mathbb{R}$ **(respectively,** $c \in \mathbb{C}$**) and also fix an arbitrary** R **with** $R \in (0, \infty)$ **or** $R = \infty$**. The** *open disc in* $\mathbb{R}$**, with** *center* c **and** *radius* R **is simply** *the open interval*

$$D(c, R) := (c - R, c + R) := \{x \in \mathbb{R} : c - R < x < c + R\} \text{ , } \textbf{if } R \in (0, \infty) \text{ ;}$$

and

$$(c - R, c + R) := (c - \infty, c + \infty) = (-\infty, \infty) = \mathbb{R} \text{ , } \textbf{if } R = \infty \text{ .}$$

The *open disc in* $\mathbb{C} = \mathbb{R}^2$**, with** *center* $c = \alpha + i\,\beta = (\alpha, \beta)$ **(where** $\alpha, \beta \in \mathbb{R}$**) and** *radius* R **is the** *"solid interior of a circle (excluding the circle itself)"* **given by**

$$D(c, R) := \{z \in \mathbb{C} : |z - c| < R\} \ , \ \textbf{if } R \in (0, \infty) \ ;$$

and

$$D(c, R) := \mathbb{C} \ , \ \textbf{if } R = \infty \ .$$

Definition 12.1. Let $(a_n)_{n \in \mathbb{N}_0}$ be a sequence in $\mathbb{R}$ (respectively, $\mathbb{C}$), and suppose that $c \in \mathbb{R}$ (resp. $\mathbb{C}$). Let x be a *real* (resp. *complex*) variable. We call the *sequence*

$$\left(\sum_{n=0}^{N} a_n \, (x - c)^n \right)_{N \geq 0}$$

a *power series*, with *center* c . If the *infinite sum*

$$S(x) = \sum_{n=0}^{\infty} a_n \, (x - c)^n := \lim_{N \longrightarrow \infty} \sum_{n=0}^{N} a_n \, (x - c)^n$$

exists in $\mathbb{R}$ (resp. $\mathbb{C}$), for all x in the *open disc* $D(c, R)$; for some *radius* $R \in (0, \infty]$, then we call S a *power series function* from $D(c, R)$ into $\mathbb{R}$ (resp. $\mathbb{C}$).

Example 12.2. *In an earlier lecture, we learned that*

$$\begin{aligned} S(x) &= \exp(x) := e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \dots \right] \end{aligned}$$

exists *in* $\mathbb{R}$ *(resp.* $\mathbb{C}$*), for all* $x \in \mathbb{R}$ *(resp.* $x \in \mathbb{C}$*). Let* $c = 0$ *and* $R = \infty$*. This function* $S = \exp$ *is a* **power series function** *from* $D(0, \infty) = \mathbb{R}$ *(resp.* $\mathbb{C}$*) into* $\mathbb{R}$ *(resp.* $\mathbb{C}$*). In this case,*

$$a_0 = 1 \ , \ a_1 = 1 \ , \ a_2 = \frac{1}{2!} \ , \ a_3 = \frac{1}{3!} \ , \ a_4 = \frac{1}{4!} \ ;$$

and generally,

$$a_n = \frac{1}{n!} \ , \ \text{for all } n \geq 0 \ .$$

Example 12.3. *In an earlier lecture, we learned that*

$$\begin{aligned} S(x) &= \sin(x) := \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2\,n+1}}{(2\,n+1)!} \\ &= \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n\, x^{2\,n+1}}{(2\,n+1)!} + \ldots \right] \end{aligned}$$

exists *in* $\mathbb{R}$ *(resp.* $\mathbb{C}$*), for all* $x \in \mathbb{R}$ *(resp.* $x \in \mathbb{C}$*). Let* $c = 0$ *and* $R = \infty$*. This function* $S = \sin$ *is a* **power series function** *from* $D(0, \infty) = \mathbb{R}$ *(resp.* $\mathbb{C}$*) into* $\mathbb{R}$ *(resp.* $\mathbb{C}$*). In this case,*

$$a_0 = 0\ ,\ a_1 = 1\ ,\ a_2 = 0\ ,\ a_3 = -\frac{1}{3!}\ ,\ a_4 = 0\ ,\ a_5 = \frac{1}{5!}\ ,\ a_6 = 0\ ,\ a_7 = -\frac{1}{7!}\ ;$$

and generally,

$$a_{2\,k} = 0 \quad \textit{and} \quad a_{2\,k+1} = \frac{(-1)^k}{(2\,k+1)!}\ ,\ \text{for all } k \geq 0\ .$$

Example 12.4. *Also, in an earlier lecture, we learned that*

$$S(x) = \cos(x) := \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$= \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots\right]$$

exists *in* $\mathbb{R}$ *(resp.* $\mathbb{C}$*), for all* $x \in \mathbb{R}$ *(resp.* $x \in \mathbb{C}$*). Let* $c = 0$ *and* $R = \infty$*. This function* $S = \cos$ *is a* **power series function** *from* $D(0, \infty) = \mathbb{R}$ *(resp.* $\mathbb{C}$*) into* $\mathbb{R}$ *(resp.* $\mathbb{C}$*). In this case,*

$$a_0 = 1\ ,\ a_1 = 0\ ,\ a_2 = -\frac{1}{2!}\ ,\ a_3 = 0\ ,\ a_4 = \frac{1}{4!}\ ,\ a_5 = 0\ ,\ a_6 = -\frac{1}{6!}\ ;$$

and generally,

$$a_{2k} = \frac{(-1)^k}{(2k)!} \quad \textit{and} \quad a_{2k+1} = 0\ ,\ \text{for all } k \geq 0\ .$$

Example 12.5. *Consider the Power Series*

$$\left(\sum_{n=0}^{N} x^n\right)_{N\geq 0} \quad ,$$

where x varies over all of $\mathbb{R}$ (resp. $\mathbb{C}$). For each integer $N \geq 0$ and every $x \in \mathbb{R}$ (resp. $\mathbb{C}$), we define

$$S_N(x) := \sum_{n=0}^{N} x^n = 1 + x + x^2 + x^3 + \cdots + x^N \ .$$

In this case, we clearly have that the center $c = 0$;

$$a_0 = 1 \ , \ a_1 = 1 \ , \ a_2 = 1 \ , \ a_3 = 1 \ ;$$

and generally,

$$a_n = 1 \ , \ \text{for all } n \geq 0 \ .$$

Moreover, in earlier work, we calculated that for all $x \neq 1$,

$$S_N(x) := \sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1 - x} \ .$$

Further, when $x \in \mathbb{R}$ *and* $|x| < 1$ *(i.e.,* $-1 < x < 1$*), we know that*

$$\lim_{N \longrightarrow \infty} x^{N+1} = 0 \ .$$

The argument we used to show this **readily extends** *to show that for all* **complex numbers** $x = \alpha + i\,\beta$ *(where* $\alpha, \beta \in \mathbb{R}$*), with* $|x| := \left(\alpha^2 + \beta^2\right)^{1/2} < 1$*, we also have that*

$$\lim_{N \longrightarrow \infty} x^{N+1} = 0 \ .$$

Therefore, by the **Algebra of Limits** *(A.o.L), for all* $x \in \mathbb{R}$ *(resp.* $\mathbb{C}$*) with* $|x| < 1$*, we have that*

$$\begin{aligned} S(x) &= \sum_{n=0}^{\infty} x^n := \lim_{N \longrightarrow \infty} \sum_{n=0}^{N} x^n \\ &= \lim_{N \longrightarrow \infty} S_N(x) = \lim_{N \longrightarrow \infty} \frac{1 - x^{N+1}}{1 - x} \\ &= \frac{1 - 0}{1 - x} = \frac{1}{1 - x} \ . \end{aligned}$$

In summary, for all real or complex numbers x *with* $|x| < 1$,

$$\frac{1}{1-x} = \left[1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \ldots\right] \ .$$

Note that for this **power series function**, *the center* $c = 0$ *and the radius* $R = 1$. *It can be shown that* 1 *is the largest possible radius* R *such that this* **power series** *(often called the "geometric (power) series")*

$$\left(\sum_{n=0}^{N} x^n\right)_{N \geq 0}$$

converges in $\mathbb{R}$ *(resp.* $\mathbb{C}$*) at every point of* $D(0, R)$. *[Aside: Note also that this Power Series does not converge at any point* x *on the circle of radius* 1*; i.e., where* $|x| = 1$ *].*

Consider now the following (useful) addition to the *Algebra of Limits (A.o.L.)*.

[The Super-Fact.] Let $(x_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}_0}$ be sequences in $\mathbb{R}$ (resp. $\mathbb{C}$). Let $c \in \mathbb{R}$ (resp. $\mathbb{C}$), and $R \in (0,\infty)$ or $R = \infty$.

Suppose that the *infinite sum*

$$S(x) = \sum_{n=0}^{\infty} a_n (x-c)^n := \lim_{N\longrightarrow\infty} \sum_{n=0}^{N} a_n (x-c)^n$$

exists in $\mathbb{R}$ (resp. $\mathbb{C}$), for all x in the *interval* $(c-R, c+R)$ (resp. the *open disc* $D(c,R)$).

Further suppose that

$$\lim_{n\longrightarrow\infty} x_n = L$$

exists in $(c-R, c+R)$ (resp. $D(c,R)$). Then,

$$\lim_{n\longrightarrow\infty} S(x_n) = S(L)\ .$$

Let's now use this *Super-Fact* to *calculate* the following limit. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ (resp. $\mathbb{C}$) with

$$\lim_{n\longrightarrow\infty} x_n = 0\ .$$

Also, assume $x_n \neq 0$, for all $n \in \mathbb{N}$.

Note that the expression x_n^k, used below, is defined by $x_n^k := (x_n)^k$, for all $n \in \mathbb{N}$ and for all integers $k \geq 0$. Also, concerning the notation used in the ***Super-Fact*****: note that $c = 0$ and $L = 0$.**

Problem. Using the ***Super-Fact*****, algebra, and the** ***power series formula*** **for** ***exp*****,** ***calculate***

$$\lim_{n \longrightarrow \infty} \frac{\exp(x_n) - 1}{x_n} .$$

Solution. Fix an arbitrary $n \in \mathbb{N}$. Note that $x_n \neq 0$.

$$\begin{aligned} \frac{\exp(x_n) - 1}{x_n} &= \frac{1}{x_n} \left(\exp(x_n) - 1\right) \\ &= \frac{1}{x_n} \left(\sum_{k=0}^{\infty} \frac{x_n^k}{k!} - 1 \right) \\ &= \frac{1}{x_n} \left(\left[1 + x_n + \frac{x_n^2}{2!} + \frac{x_n^3}{3!} + \frac{x_n^4}{4!} + \cdots + \frac{x_n^k}{k!} + \ldots \right] - 1 \right) . \end{aligned}$$

Thus, by the *distributive law*,

$$\begin{aligned}\frac{\exp(x_n)-1}{x_n} &= \frac{1}{x_n}\left(x_n+\frac{x_n^2}{2!}+\frac{x_n^3}{3!}+\frac{x_n^4}{4!}+\cdots+\frac{x_n^k}{k!}+\dots\right)\\ &= \frac{1}{x_n}x_n\left(1+\frac{x_n}{2!}+\frac{x_n^2}{3!}+\frac{x_n^3}{4!}+\cdots+\frac{x_n^{k-1}}{k!}+\dots\right)\\ &= \left[1+\frac{x_n}{2!}+\frac{x_n^2}{3!}+\frac{x_n^3}{4!}+\cdots+\frac{x_n^{k-1}}{k!}+\dots\right]\\ &= S(x_n)\ ,\end{aligned}$$

where S is the *power series function* defined by

$$S(x) := \left[1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots+\frac{x^{k-1}}{k!}+\dots\right]\ .$$

Note that the *infinite sum $S(x)$ exists* in $\mathbb{R}$ (resp. $\mathbb{C}$), because $\exp(x)$ does, for all $x \in \mathbb{R}$ (resp. $\mathbb{C}$).

By the *Super-Fact***,**

$$\begin{aligned}\lim_{n\longrightarrow\infty}\frac{\exp(x_n)-1}{x_n} &= \lim_{n\longrightarrow\infty} S(x_n) = S(0)\\ &= \left[1+\frac{0}{2!}+\frac{0^2}{3!}+\frac{0^3}{4!}+\cdots+\frac{0^{k-1}}{k!}\ldots\right]\\ &:= \lim_{N\longrightarrow\infty}\left[1+\frac{0}{2!}+\frac{0^2}{3!}+\frac{0^3}{4!}+\cdots+\frac{0^{N-1}}{N!}\right]\\ &= \lim_{N\longrightarrow\infty}[1] = 1\ .\end{aligned}$$

13. Math 0235 : Lecture 13 : Wed 24/September/08

Homework Assignment 13 [HA.L13]

We will abbreviate the word "respectively" to *"resp."***.**

[1] Recall from Lecture 12 (Monday, 22/Sep/08) the following addition to the *Algebra of Limits (A.o.L.)***:**

[The Super-Fact.] Let $(x_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}_0}$ be sequences in $\mathbb{R}$ (resp. $\mathbb{C}$). Let $c \in \mathbb{R}$ (resp. $\mathbb{C}$), and $R \in (0,\infty)$ or $R = \infty$.

Suppose that the *infinite sum*

$$S(x) = \sum_{n=0}^{\infty} a_n \,(x-c)^n := \lim_{N\longrightarrow\infty} \sum_{n=0}^{N} a_n \,(x-c)^n$$

exists in $\mathbb{R}$ (resp. $\mathbb{C}$), for all x in the *interval* $(c-R, c+R)$ (resp. the *open disc* $D(c,R)$).

Further suppose that

$$\lim_{n\longrightarrow\infty} x_n = L$$

exists in $(c-R, c+R)$ (resp. $D(c,R)$). Then,

$$\lim_{n\longrightarrow\infty} S(x_n) = S(L)\ .$$

Remark: You may have encountered this idea before... It is one way of saying that the function S is continuous (on its given interval (resp. disc) domain).

Here, recall that $(c - R, c + R) := \{x \in \mathbb{R} : c - R < x < c + R\}$**, if** $R \in (0, \infty)$**; and** $(c - R, c + R) = \mathbb{R}$**, if** $R = \infty$**. Also recall that** $D(c, R) := \{z \in \mathbb{C} : |z - c| < R\}$**, if** $R \in (0, \infty)$**; and** $D(c, R) = \mathbb{C}$**, if** $R = \infty$**. Note that for the** *power series function* S **defined above, the value of** S **at the** *center* c **is** $S(c) = a_0$ **.**

Also, note that the expression x_n^k**, used in the problems below, is defined by** $x_n^k := (x_n)^k$**, for all** $n \in \mathbb{N}$ **and for all integers** $k \geq 0$**.**

In the problems below, $c = 0$ **and** $L = 0$**. In particular, we are assuming that**

$$\lim_{n \longrightarrow \infty} x_n = 0 \ .$$

Also, assume $x_n \neq 0$**, for all** $n \in \mathbb{N}$**.**

Using the *A.o.L. Super-Fact*, *algebra*, **and known ways of expressing some of the functions below as** *sums of certain Power Series*, **solve the following 3 problems.**

Problem(1). *Calculate*

$$\lim_{n\longrightarrow\infty} \frac{\sin(x_n) - x_n + \dfrac{x_n^3}{6}}{x_n^5} \ .$$

Problem(2). *Calculate*

$$\lim_{n\longrightarrow\infty} \frac{\exp(x_n) - 1 - x_n - \dfrac{x_n^2}{2}}{x_n^3} \ .$$

Problem(3). *Calculate*

$$\lim_{n\longrightarrow\infty} \frac{\exp(x_n) - \dfrac{1}{1 - x_n}}{x_n^2} \ .$$

Next we will solve some *example problems*...

Example Problem(1). *Calculate*

$$\lim_{n\longrightarrow\infty} \frac{\sin(x_n)}{x_n} \ .$$

Solution. Fix an arbitrary $n \in \mathbb{N}$. Note that $x_n \neq 0$.

$$\begin{aligned}
\frac{\sin(x_n)}{x_n} &= \frac{1}{x_n}\,\sin(x_n) \\
&= \frac{1}{x_n}\,\sum_{k=0}^{\infty} \frac{(-1)^k\,x_n^{2\,k+1}}{(2\,k+1)!} \\
&= \frac{1}{x_n}\,\left[x_n - \frac{x_n^3}{3!} + \frac{x_n^5}{5!} - \frac{x_n^7}{7!} + \cdots + \frac{(-1)^k\,x_n^{2\,k+1}}{(2\,k+1)!} + \ldots\right] \;.
\end{aligned}$$

Thus, by the *distributive law***,**

$$\begin{aligned}
\frac{\sin(x_n)}{x_n} &= \frac{1}{x_n}\,x_n\,\left[1 - \frac{x_n^2}{3!} + \frac{x_n^4}{5!} - \frac{x_n^6}{7!} + \cdots + \frac{(-1)^k\,x_n^{2\,k}}{(2\,k+1)!} + \ldots\right] \\
&= \left[1 - \frac{x_n^2}{3!} + \frac{x_n^4}{5!} - \frac{x_n^6}{7!} + \cdots + \frac{(-1)^k\,x_n^{2\,k}}{(2\,k+1)!} + \ldots\right] \\
&= S(x_n)\;,
\end{aligned}$$

where S is the *power series function* **defined by**

$$S(x) := \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^k\,x^{2\,k}}{(2\,k+1)!} + \ldots\right] \;.$$

Note that the *infinite sum* $S(x)$ *exists* **in** $\mathbb{R}$ **(resp.** $\mathbb{C}$**), because** $\sin(x)$ **does, for all** $x \in \mathbb{R}$ **(resp.** $\mathbb{C}$**).**

By the *Super-Fact*, **since** $\lim_{n\longrightarrow\infty} x_n = 0$**,**

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \frac{\sin(x_n)}{x_n} &= \lim_{n\longrightarrow\infty} S(x_n) = S(0) \\
&= \left[1 - \frac{0^2}{3!} + \frac{0^4}{5!} - \frac{0^6}{7!} + \cdots + \frac{(-1)^k\, 0^{2\,k}}{(2\,k+1)!} + \ldots\right] \\
&:= \lim_{N\longrightarrow\infty} \left[1 - \frac{0^2}{3!} + \frac{0^4}{5!} - \frac{0^6}{7!} + \cdots + \frac{(-1)^N\, 0^{2\,N}}{(2\,N+1)!}\right] \\
&= \lim_{N\longrightarrow\infty} [1] = 1 \ .
\end{aligned}$$

Example Problem(2). *Calculate*

$$\lim_{n\longrightarrow\infty} \frac{\cos(x_n) - 1}{x_n^2} \ .$$

Solution. Fix an arbitrary $n \in \mathbb{N}$. Note that $x_n \neq 0$.

$$\begin{aligned}
\frac{\cos(x_n) - 1}{x_n^2} &= \frac{1}{x_n^2}\,(\cos(x_n) - 1) \\
&= \frac{1}{x_n^2}\left(\sum_{k=0}^{\infty} \frac{(-1)^k\, x_n^{2\,k}}{(2\,k)!} - 1\right) \\
&= \frac{1}{x_n^2}\left(\left[1 - \frac{x_n^2}{2!} + \frac{x_n^4}{4!} - \frac{x_n^6}{6!} + \cdots + \frac{(-1)^k\, x_n^{2\,k}}{(2\,k)!} + \ldots\right] - 1\right) \;.
\end{aligned}$$

Thus, by the *distributive law***,**

$$\begin{aligned}
\frac{\cos(x_n) - 1}{x_n^2} &= \frac{1}{x_n^2}\left(-\frac{x_n^2}{2!} + \frac{x_n^4}{4!} - \frac{x_n^6}{6!} + \cdots + \frac{(-1)^k\, x_n^{2\,k}}{(2\,k)!} + \ldots\right) \\
&= \frac{1}{x_n^2}\, x_n^2 \left(-\frac{1}{2!} + \frac{x_n^2}{4!} - \frac{x_n^4}{6!} + \cdots + \frac{(-1)^k\, x_n^{2\,k-2}}{(2\,k)!} + \ldots\right) \\
&= \left[-\frac{1}{2!} + \frac{x_n^2}{4!} - \frac{x_n^4}{6!} + \cdots + \frac{(-1)^k\, x_n^{2\,k-2}}{(2\,k)!} + \ldots\right] \\
&= S(x_n) \;,
\end{aligned}$$

where S **is the** *power series function* **defined by**

$$S(x) := \left[-\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \cdots + \frac{(-1)^k\, x^{2\,k-2}}{(2\,k)!} + \ldots \right] \ .$$

Note that the *infinite sum* $S(x)$ *exists* **in** $\mathbb{R}$ **(resp.** $\mathbb{C}$**), because** $\cos(x)$ **does, for all** $x \in \mathbb{R}$ **(resp.** $\mathbb{C}$**).**

By the *Super-Fact***, since** $\lim_{n \longrightarrow \infty} x_n = 0$**,**

$$\begin{aligned}
\lim_{n \longrightarrow \infty} \frac{\cos(x_n) - 1}{x_n^2} &= \lim_{n \longrightarrow \infty} S(x_n) = S(0) \\
&= \left[-\frac{1}{2!} + \frac{0^2}{4!} - \frac{0^4}{6!} + \cdots + \frac{(-1)^k\, 0^{2\,k-2}}{(2\,k)!} + \ldots \right] \\
&:= \lim_{N \longrightarrow \infty} \left[-\frac{1}{2!} + \frac{0^2}{4!} - \frac{0^4}{6!} + \cdots + \frac{(-1)^N\, 0^{2\,N-2}}{(2\,N)!} \right] \\
&= \lim_{N \longrightarrow \infty} \left[-\frac{1}{2!} \right] = -\frac{1}{2} \ .
\end{aligned}$$

Example Problem(3). *Calculate*

$$\lim_{n \longrightarrow \infty} \frac{\cos(x_n) - 1}{x_n} \ .$$

Solution. We can solve this problem similarly to Example Problem (2)... Or we can use the following *shortcut*. **By Example Problem(2),**

$$\lim_{n \longrightarrow \infty} \frac{\cos(x_n) - 1}{x_n^2} = -\frac{1}{2} \ .$$

For each $n \in \mathbb{N}$**,**

$$\frac{\cos(x_n) - 1}{x_n} = x_n \left(\frac{\cos(x_n) - 1}{x_n^2} \right) \ .$$

Recall that

$$\lim_{n \longrightarrow \infty} x_n = 0 \ .$$

Thus, from the *Algebra of Limits***,**

$$\lim_{n \longrightarrow \infty} \frac{\cos(x_n) - 1}{x_n} = (0) \left(-\frac{1}{2} \right) = 0 \ .$$

Homework Assignment 14 [HA.L14]

[1] Carefully read and note Section 9.2 of our text *[Stewart]* **on "Vectors", pages 642-649. Then** *do* **problems 4, 5, 6, 7, 10, 11, 14, 15, 18, 19, 20, 24, 26, 34, 37, 38 on pages 649-651.**

[2] Fix an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ (resp. $\mathbb{C}$). Let $L \in \mathbb{R}$ (resp. $\mathbb{C}$). Suppose that

$$\lim_{n\longrightarrow\infty} x_n = L \ ; \ \textbf{and} \ \ x_n \neq L \ , \ \textbf{for all } n \in \mathbb{N} \ .$$

Using the *A.o.L. Super-Fact*, *algebra*, **and known ways of expressing some of the functions below as** *infinite sums*, **solve the following 3 problems.**

Problem(1). Suppose that $L = 0$. *Calculate*

$$\lim_{n\longrightarrow\infty} \frac{\left(\exp(x_n) - \dfrac{\left(1 + \dfrac{x_n}{2}\right)}{\left(1 - \dfrac{x_n}{2}\right)}\right)}{x_n^3} \ .$$

Problem(2). Suppose that $L = \pi/2$**.** *Calculate*

$$\lim_{n \longrightarrow \infty} \frac{\left(\sin(x_n) - 1 + \frac{1}{2}\left(x_n - \frac{\pi}{2}\right)^2\right)}{\left(x_n - \frac{\pi}{2}\right)^4} .$$

[**Hint for solving Problem (2): Also use** *trigonometry*...]

Problem(3). Suppose that $L = 0$**.** *Calculate*

$$\lim_{n \longrightarrow \infty} \frac{\tan(x_n) - x_n - \dfrac{x_n^3}{3}}{x_n^5} .$$

Hint for Problem(3):

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \ , \ \textbf{for all } x \in \mathbb{R} \textbf{ with } \cos(x) \neq 0 .$$

End of [HA.L14].

We introduce here an extension of the *Algebra of Limits***.**

Fact 14.1 (Part of the Algebra of Limits (A.o.L.)). *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{R}$, $L \in \mathbb{R}$ *and* $w \in \mathbb{R}$. *Suppose that*

$$\lim_{n\longrightarrow\infty} a_n = L \ .$$

(1) *If there exists* $k \in \mathbb{N}$ *such that*

$$a_n \geq w \ , \ \textit{for all integers } n \geq k \ ,$$

then

$$L \geq w \ .$$

(2) *Also, if there exists* $k \in \mathbb{N}$ *such that*

$$a_n \leq w \ , \ \textit{for all integers } n \geq k \ ,$$

then

$$L \leq w \ .$$

We see that Fact 14.1 is related to *the Squeeze Theorem*. **Note that there is** *no* **A.o.L. analogue of this fact for general sequences of** *complex numbers* $(a_n)_{n\in\mathbb{N}}$**, because this fact deals with the natural** *ordering* $\leq$ **on** $\mathbb{R}$. *Indeed, note that all of the parts of the A.o.L. stated for real number sequences, still work for complex number sequences:* **except for** *those parts that concern the ordering* $\leq$ *on* $\mathbb{R}$.

Recall the following useful fact:
(#) [For all $u, v > 0$**, the** *product* $u\,v > 0$**.]**

Theorem 14.2. (1) *For all* $x \in \mathbb{R}$ *with* $x > 0$,

$$e^x > 1 \ .$$

(2) *For all* $x \in \mathbb{R}$,

$$e^x > 0 \ .$$

(3) *For all* $x, z \in \mathbb{R}$ *with* $x < z$,

$$e^x < e^z \ ;$$

i.e., the exponential function exp *is strictly increasing on* $\mathbb{R}$.

Proof. (1) Fix an arbitrary $x \in \mathbb{R}$ with $x > 0$.

$$e^x = \exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{N \longrightarrow \infty} \sum_{n=0}^{N} \frac{x^n}{n!}$$

Fix an arbitrary $N \in \mathbb{N}$ with $N \geq 2$, and define

$$S_N := \sum_{n=0}^{N} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^N}{N!} \ .$$

By Fact(#) above, since $x > 0$, we see that $x^2 := x\,x > 0$. Further, $x^3 := x\,x^2 > 0$; and inductively we see that $x^k := x\,x^{k-1} > 0$, for all integers $k \geq 2$. [Aside: Of course, for any real number $x \neq 0$, we have that $x^2 > 0, x^4 > 0, x^6 > 0$, etc... However, the *odd powers* of x are only *positive* when x itself is positive...]

Since sums of positive numbers are also *positive*, we have that

$$\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^N}{N!} > 0 \ ,$$

and consequently,

$$\begin{aligned} S_N &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^N}{N!} \\ &> 1 + x + 0 = 1 + x \ . \end{aligned}$$

In summary, for all integers $N \geq 2$,

$$S_N \geq 1 + x \ .$$

By Fact 14.1 part (1),

$$e^x := \lim_{N \longrightarrow \infty} S_N \geq 1 + x > 1 + 0 = 1 \ .$$

Hence, for all real $x > 0$, $e^x > 1$.

(2) Fix an arbitrary $x \in \mathbb{R}$.
[Case 1. $x > 0$.] By Part (1), $e^x > 1 > 0$; and so $e^x > 0$.
[Case 2. $x = 0$.] Clearly, $e^0 = \exp(0) = 1 > 0$. So, $e^0 > 0$.
[Case 3. $x < 0$.] By the addition formula for exp,

$$e^x \, e^{-x} = e^{x+(-x)} = e^0 = 1 \; .$$

Recall that for every real number $u > 0$, we have that its *reciprocal* $1/u$ satisfies

$$\frac{1}{u} > 0 \; .$$

But $-x > 0$, and so from Case 1 above, $e^{-x} > 0$. Hence,

$$e^x = \frac{1}{e^{-x}} > 0 \; .$$

In summary, we have shown that $e^x > 0$.

(3) Fix arbitrary $x, z \in \mathbb{R}$ with $x < z$. Thus, $z - x > 0$.
Further,

$$\begin{aligned} e^z - e^x &= e^{x+(z-x)} - e^x = e^x \, e^{z-x} - e^x \\ &= e^x \, (e^{z-x} - 1) \; . \end{aligned}$$

Now, from Part (2), $u := e^x > 0$. Also, $z - x > 0$; and therefore Part (1) gives us that $e^{z-x} > 1$. Hence, $v := e^{z-x} - 1 > 0$. Applying Fact (#) above, we conclude that

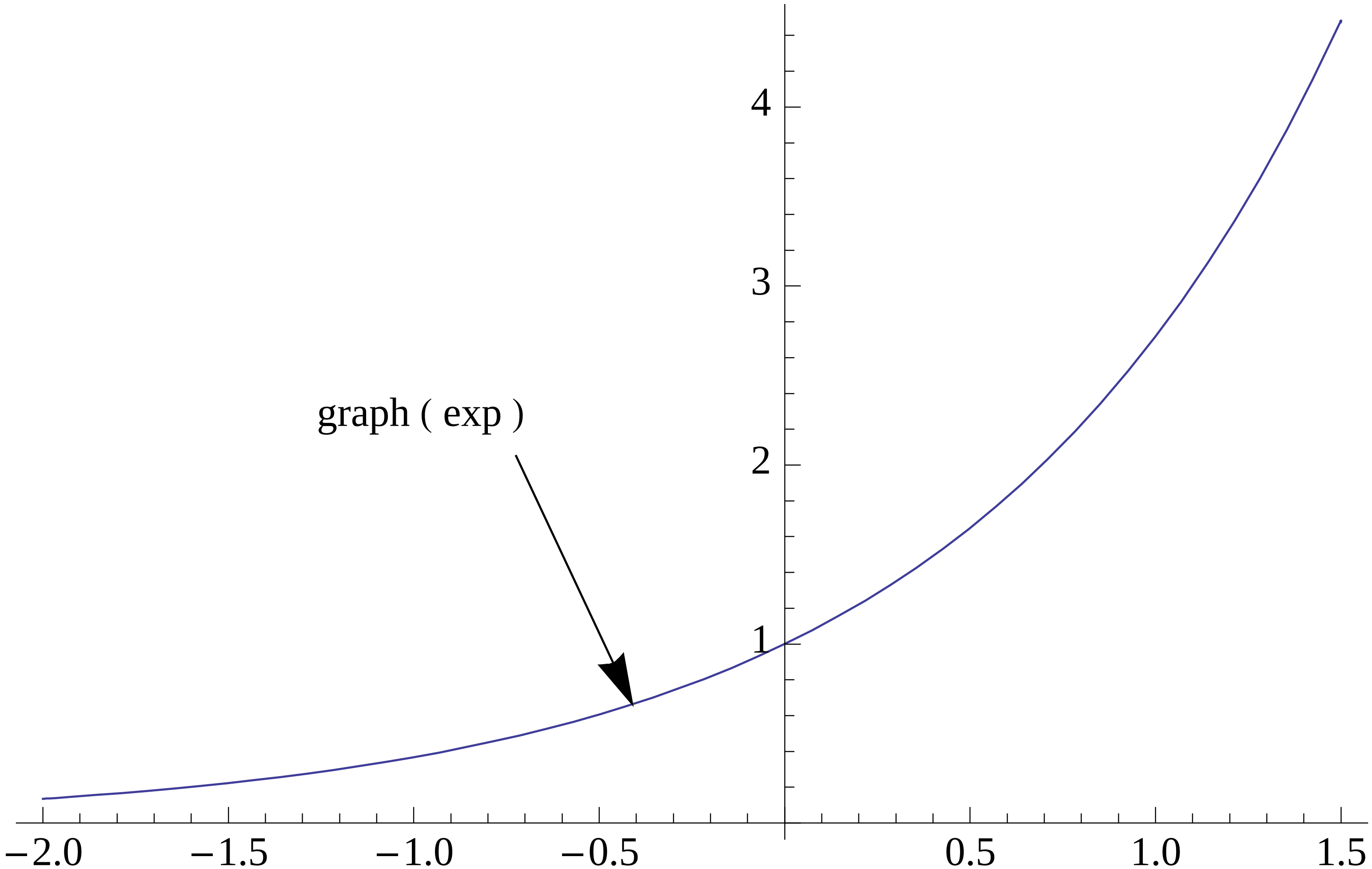
$$e^z - e^x = e^x\,(e^{z-x} - 1) = u\,v > 0\ .$$

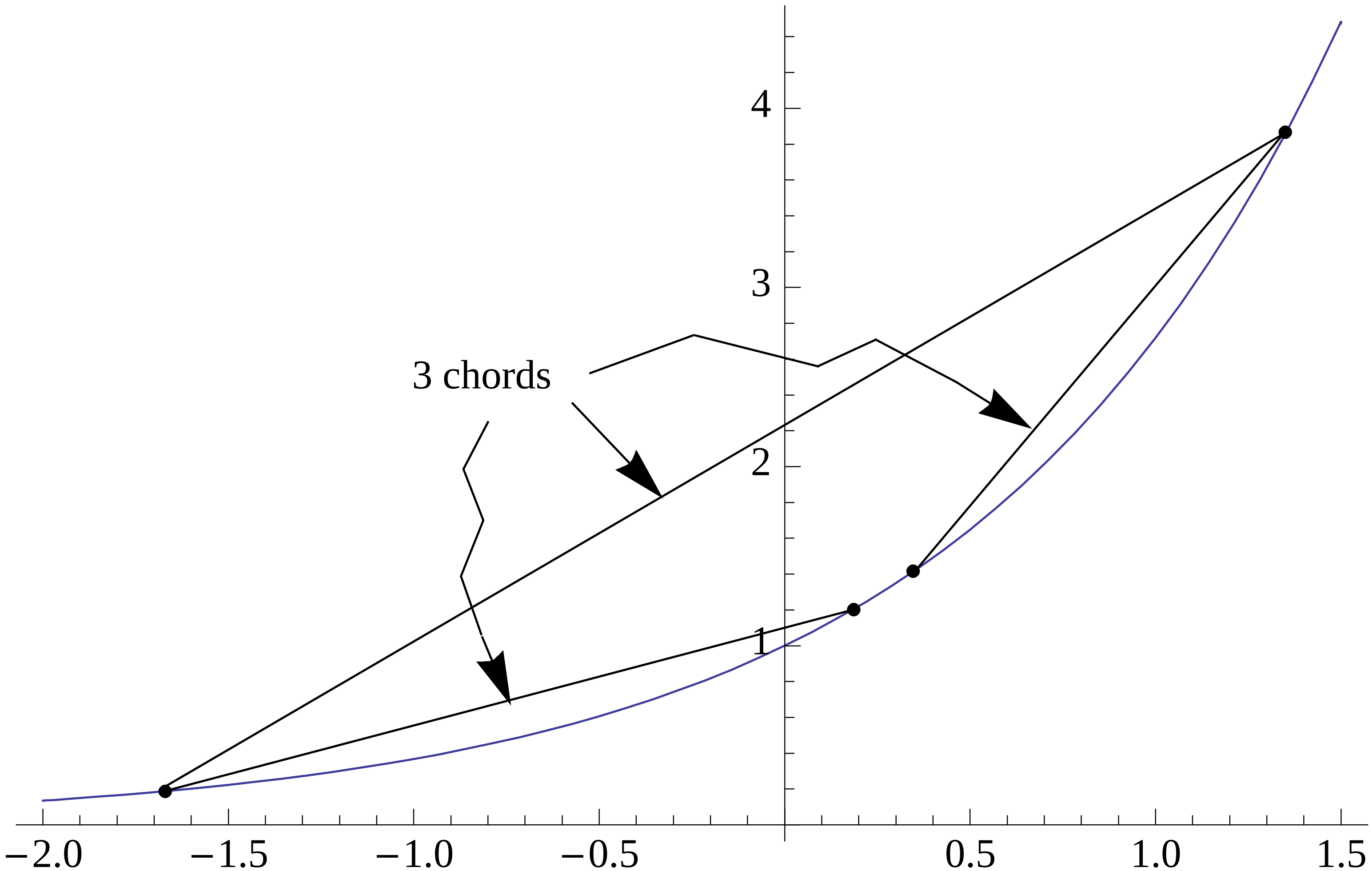
In summary, $e^z - e^x > 0$, and consequently $e^z > e^x$; i.e., $e^x < e^z$. $\square$

Theorem 14.2 enables us to roughly sketch the graph of exp *near* 0 **... The first graph below clearly illustrates all three parts of Theorem 14.2. As we will see later (and as you may have learned before), there are other aspects of the graph of** exp **that Theorem 14.2** *does not* **tell us... In particular,** exp **is a function that is** *concave up*; **i.e.,** *every straight line segment joining two points on the graph (that is, every "chord") lies above or on the graph*. [**See the second graph below...**] **It turns out that** *the second derivative* **of** exp **is always positive, which implies that** exp **is concave up... Also, what does the graph of** exp **"do" when** x **is** *extremely large and positive* **or** x **is** *very large in magnitude and negative?* **The following**

limits, **that we will discuss in more detail later, give us an answer:**

$$\lim_{x \longrightarrow \infty} e^x = \infty \quad \textbf{and} \quad \lim_{x \longrightarrow -\infty} e^x = 0 \ .$$





4
3
2
1
3 chords
−2.0
−1.5
−1.0
−0.5
0.5
1.0
1.5

"More thoughts related to our earlier discussion of *true-or-false logical sentences*...**"**

Recall from an earlier lecture the following "nonsense question and answer":

Why is a duck ?

Answer: Because one of its legs are both the same.

During Lecture 14, I suggested that students Google for "an answer" to the following question... :

Why is a mouse when it spins?

A common answer on the web seems to be:

Answer: The higher, the fewer.

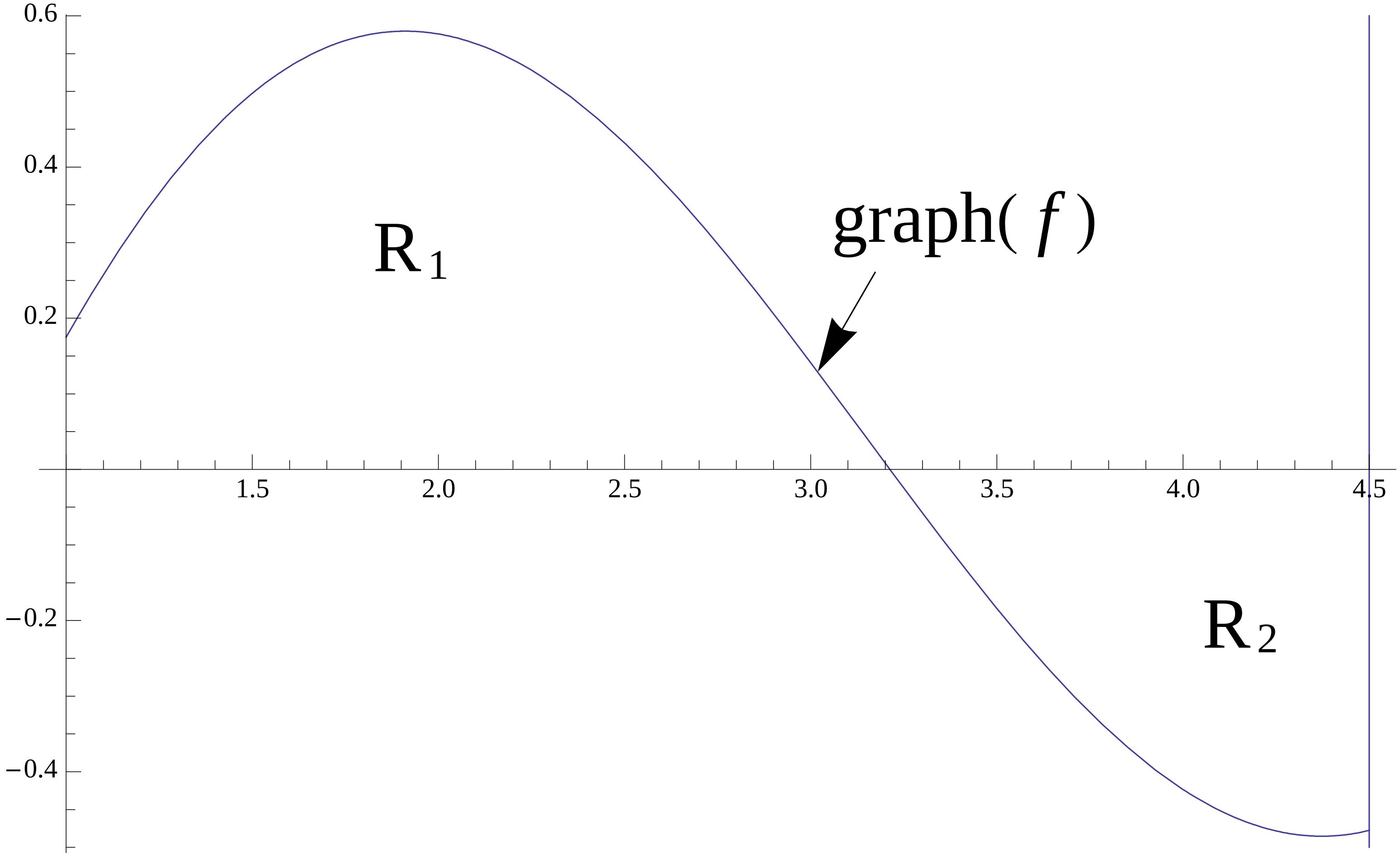
15. Math 0235 : Lecture 15 : Mon 29/September/08

Fix arbitrary real numbers a **and** b **with** $a < b$**. Recall that the** *closed and bounded interval* $[a, b]$ **is given by**

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\} \ .$$

Consider an arbitrary function $f : [a, b] \longrightarrow \mathbb{R}$**. We wish to** *define* **the real number "**$I :=$ *the integral of* f **over** $[a, b]$**",** *if it exists...*

Consider the following picture: *Graph 1*



The *idea* **is that** I *should be given by*:

$$I = \text{“}\mathbf{Area}(\mathbf{Region}\ R_1) - \mathbf{Area}(\mathbf{Region}\ R_2)\text{”}\ .$$

Put differently, I **should equal** *"the signed (or net) area between* **graph**(f) *and the horizontal or x-axis (i.e., the interval $[a, b]$)."*

In an attempt to *carefully define* **"**$I :=$ *the integral of* f **over** $[a, b]$**"** **in such a way that** *"a large and useful collection of functions have an integral I"*, **we next make some helpful definitions...**

Definition 15.1. (1) We define the set $\mathcal{P}[a, b]$ of *all partitions* $\overrightarrow{P} = (x_0, x_1, \ldots, x_n)$ of $[a, b]$ by

$$\mathcal{P}[a,b] := \Big\{ \text{finite sequences } \overrightarrow{P} = (x_0, x_1, \ldots, x_n) : \\ n \in \mathbb{N} \ , \ \text{each } x_j \in \mathbb{R} \ , \ \text{and } a = x_0 < x_1 < \cdots < x_n = b \Big\} \ .$$

(2) Fix an arbitrary *partition* $\overrightarrow{P} = (x_0, x_1, \ldots, x_n) \in \mathcal{P}[a, b]$. A *finite sequence* $\overrightarrow{t} = (t_1, \ldots, t_n)$ is called a *tag sequence for* $\overrightarrow{P}$ if

$$t_k \in [x_{k-1}, x_k] \ , \ \text{for all } k \in \{1, \ldots, n\} \ .$$

We denote *the set of all tag sequences* $\overrightarrow{t}$ *for* $\overrightarrow{P}$ by

$$\operatorname{tag}(\overrightarrow{P}) \ .$$

Two interesting examples of *tag sequences* are $\overrightarrow{t}_{\text{left}}$ and $\overrightarrow{t}_{\text{right}}$, defined by:

$$\overrightarrow{t}_{\text{left}} := (x_0, \ldots, x_{n-1}) \quad \text{and} \quad \overrightarrow{t}_{\text{right}} := (x_1, \ldots, x_n) \ .$$

(3) Fix an arbitrary $\overrightarrow{P} \in \mathcal{P}[a, b]$. Next fix an arbitrary $\overrightarrow{t} \in \text{tag}(\overrightarrow{P})$. We define the $(\overrightarrow{P}, \overrightarrow{t})$-*Riemann sum*, $S(f, \overrightarrow{P}, \overrightarrow{t})$, by

$$S(f, \overrightarrow{P}, \overrightarrow{t}) := \sum_{k=1}^{n} f(t_k)\,(x_k - x_{k-1}) \ .$$

***** Note concerning Definition 15.1 (3): Fix an arbitrary $k \in \{1, \ldots, n\}$.**

If $f(t_k) > 0$, then:
$f(t_k)\,(x_k - x_{k-1})$ = *[the area of the shaded rectangle* **above** *the interval $[x_{k-1}, x_k]$ in the picture "Graph 2" below].*

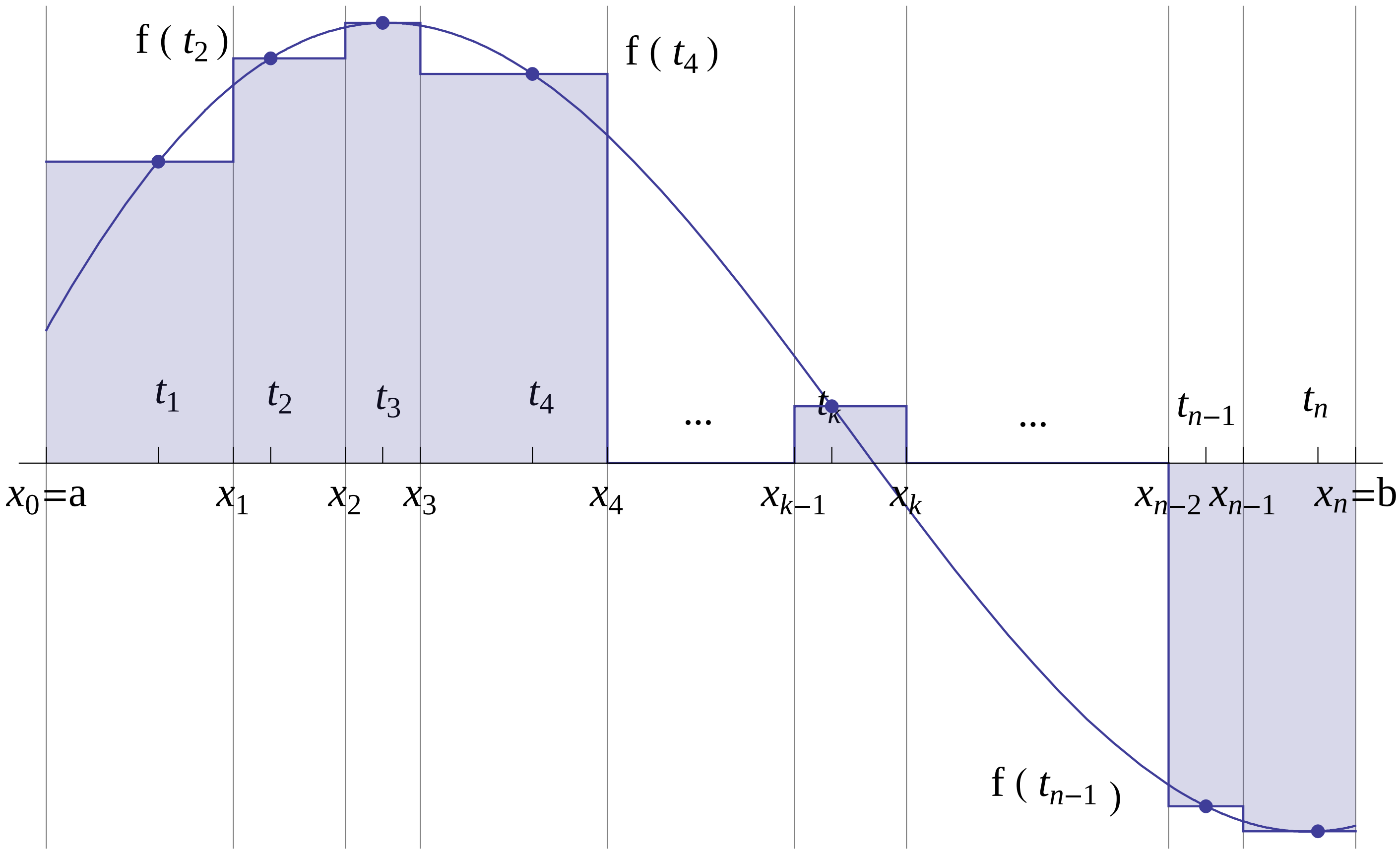
If $f(t_k) < 0$, then:
$f(t_k)\,(x_k - x_{k-1})$ = **-** *[the area of the shaded rectangle* **below** *the interval $[x_{k-1}, x_k]$ in the picture "Graph 2" below].*

Thus, the $(\overrightarrow{P}, \overrightarrow{t})$**-***Riemann sum*

$$S(f, \overrightarrow{P}, \overrightarrow{t}) := \sum_{k=1}^{n} f(t_k)\,(x_k - x_{k-1})$$

is *[the signed area between* **the shaded bar graph in** *"Graph 2"*, **drawn below,** *and the horizontal axis].*

Graph 2:



For any non-empty, finite set of real numbers $S := \{s_1, \ldots, s_m\}$**, we define** $\max S$ **to be the** *maximum value* **among all of the numbers** $s_1, \ldots, s_m$**. E.g.,** $\max\{4, -3, 2, 1\} = 4$**.**

For all partitions $\overrightarrow{P} \in \mathcal{P}[a,b]$, we define the *mesh-size* **of $\overrightarrow{P}$,** $\mathrm{mesh}(\overrightarrow{P})$ **by**

$$\mathrm{mesh}(\overrightarrow{P}) := \max\left\{x_k - x_{k-1} : k \in \{1, \ldots, n\}\right\} .$$

The *key idea* **underlying (Riemann)** *integration* **is that as we vary $\overrightarrow{P}$ over** $\mathcal{P}[a,b]$ **and also vary $\overrightarrow{t}$ over** $\mathrm{tag}(\overrightarrow{P})$ *in such a way that*

$$\mathrm{mesh}(\overrightarrow{P}) \longrightarrow 0 \ , \ \textbf{and simultaneously } n \longrightarrow \infty \ ,$$

the *Riemann sums*

$$S(f, \overrightarrow{P}, \overrightarrow{t}) := \sum_{k=1}^{n} f(t_k)\,(x_k - x_{k-1})$$

should converge **to**

$I = [$**the signed area between** $\mathrm{graph}(f)$ **and the horizontal axis**$]$,

if this **limit** *exists in* $\mathbb{R}$**...**

16. Math 0235 : Lecture 16 : Wed 1/October/08

Definition 16.1. Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Let $I \in \mathbb{R}$. We say that *"f is Riemann integrable over* $[a, b]$*"* with *integral* I if:
for all $\varepsilon > 0$, there exists $\delta > 0$ so that for all *partitions* $\overrightarrow{P} \in \mathcal{P}[a, b]$ with $\mathrm{mesh}(\overrightarrow{P}) < \delta$, for all *tag sequences* $\overrightarrow{t} \in \mathrm{tag}(\overrightarrow{P})$,

$$\left| I - S(f, \overrightarrow{P}, \overrightarrow{t}) \right| < \varepsilon \ .$$

In this situation, we write

$$I = \int_a^b f(x)\, dx := \lim_{\mathrm{mesh}(\overrightarrow{P}) \longrightarrow 0} S(f, \overrightarrow{P}, \overrightarrow{t}) \ .$$

We also denote the *set of all Riemann-integrable functions* $f : [a, b] \longrightarrow \mathbb{R}$ by

$$\mathcal{R}[a, b] \ .$$

Notes about Definition 16.1: [1] Recall that

$$S(f, \overrightarrow{P}, \overrightarrow{t}) := \sum_{k=1}^{n} f(t_k)\,(x_k - x_{k-1}) \ .$$

[2] We also denote $I = \int_a^b f(x)\,dx$ **by**

$$I = \int_{x=a}^{x=b} f(x)\,dx = \int_{u=a}^{u=b} f(u)\,du = \int_a^b f(u)\,du \ .$$

[3] In the above integrals, x **and** u **are (interchangeable)** *dummy variables***.**

Definition 16.2. Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function.
(1) We say that *"f is increasing on $[a, b]$"* if

$$\Big[\text{ for all } x < z \text{ in } [a, b]\ ,\ f(x) \le f(z) \ \Big] \ .$$

(2) We say that *"f is decreasing on $[a, b]$"* if

$$\Big[\text{ for all } x < z \text{ in } [a, b]\ ,\ f(x) \ge f(z) \ \Big] \ .$$

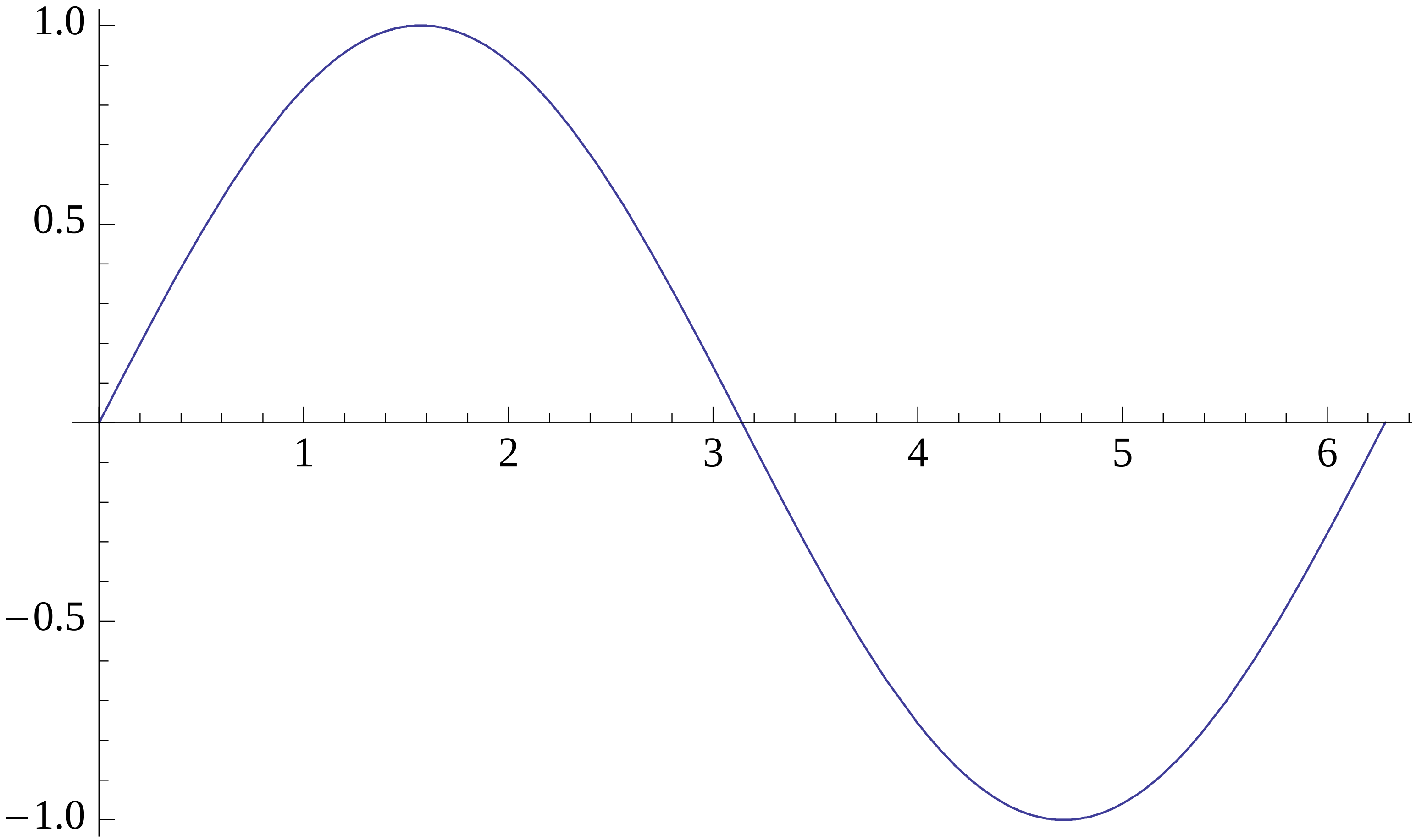
(3) We say that *"f is monotone on $[a,b]$"* if

$$\Big[\text{ either } f \textit{ is increasing on } [a,b] \text{ or } f \textit{ is decreasing on } [a,b] \ \Big].$$

(4) We say that *"f is piecewise monotone on $[a,b]$"* if there exists a partition $\overrightarrow{Q} = (u_0, \ldots, u_m)$ of $[a,b]$ such that

$$\Big[\text{ for all } k \in \{1, \ldots, m\}\ ,\ f \text{ is monotone on } [u_{k-1}, u_k]\ \Big]\ .$$

Notes about Definition 16.2: [1] If the function f is monotone on $[a,b]$, then f is piecewise monotone on $[a,b]$. (Let $u_0 = a$ and $u_1 = b$...)
[2] The functions $\exp$, and $[f(x) := x^n$, where n is an odd positive integer constant$]$ are *increasing* functions on every interval $[a,b]$.
[3] The functions $[f(x) := x^n$, where n is a negative integer constant$]$ are *decreasing* functions on every interval $[a,b]$ with $a > 0$.
[4] The functions $\sin$, $\cos$ and $[f(x) := x^n$, where n is an even positive integer constant$]$ are *piecewise monotone* functions on every interval $[a,b]$.
[5] E.g., for the function $[\sin : [0, 2\pi] \longrightarrow \mathbb{R}]$, let $u_0 := 0$, $u_1 := \pi/2$, $u_2 := 3\pi/2$ and $u_3 := 2\pi$. Then $\overrightarrow{Q} = (u_0, u_1, u_2, u_3)$ is a partition of $[0, 2\pi]$ such that $\sin$ is *monotone* on each interval $[u_{k-1}, u_k]$.



[⚡⚡]["**Double-lightning-bolt**"]**Theorem** *Let* $a, b \in \mathbb{R}$ *with* $a < b$. *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be a* **piecewise monotone** *function. Then* f *is* **Riemann integrable on** $[a, b]$.

For a proof: **See future courses: e.g., Math 0420, 0450 and/or 1530; or certain books in the Math Library...**

So, all the *piecewise monotone functions* **noted above:** exp**,** sin**,** cos**, etc, are** *Riemann integrable***...**

Note that when $f \in \mathcal{R}[a,b]$**, we can** *calculate* $I = \int_a^b f(x)\,dx$ **in the following way.**

Consider any sequence of *partitions* $(\overrightarrow{P}^{(n)})_{n\in\mathbb{N}}$ **with**

$$\lim_{n\longrightarrow\infty} \mathrm{mesh}(\overrightarrow{P}^{(n)}) = 0 \ ,$$

and any sequence of *tag sequences* $(\overrightarrow{t}^{(n)})_{n\in\mathbb{N}}$ **with each**

$$\overrightarrow{t}^{(n)} \in \mathrm{tag}(\overrightarrow{P}^{(n)}) \ .$$

Then we have

$$\int_a^b f(x)\,dx = \lim_{n\longrightarrow\infty} S(f, \overrightarrow{P}^{(n)}, \overrightarrow{t}^{(n)}) \ .$$

For example, consider the sequence of *equal width partitions* $(\overrightarrow{P}^{(n)})_{n\in\mathbb{N}}$ **and the corresponding sequence of** *right tag sequences* $(\overrightarrow{t}^{(n)}_{\mathrm{right}})_{n\in\mathbb{N}}$ **defined in this way.**

Fix an arbitrary integer $n \in \mathbb{N}$. Define

$$\delta_n := \frac{b-a}{n} \ .$$

Next, define

$$P^{(n)} := \left(x_0^{(n)}, \ldots, x_k^{(n)}, \ldots, x_n^{(n)} \right) \ ,$$

where for all $k \in \{0, 1, \ldots, n\}$,

$$x_k^{(n)} := a + k\,\delta_n \ .$$

In particular, $x_0^{(n)} = a$, $x_1^{(n)} = a + \delta_n$ and

$$x_n^{(n)} = a + n\,\delta_n = a + n\left(\frac{b-a}{n}\right) = a + (b-a) = b \ .$$

Hence, $a = x_0^{(n)} < x_1^{(n)} < \cdots < x_n^{(n)} = b$. Also, for all $k \in \{1, \ldots, n\}$,

$$x_k^{(n)} - x_{k-1}^{(n)} = (a + k\,\delta_n) - (a + (k-1)\,\delta_n) = \delta_n \ .$$

It follows that

$$\mathrm{mesh}(\overrightarrow{P}^{(n)}) = \max\left\{ x_k^{(n)} - x_{k-1}^{(n)} : k \in \{1, \ldots, n\} \right\} = \delta_n := \frac{b-a}{n} \longrightarrow 0 \ ,$$

as $n \longrightarrow \infty$.

Further, for all $k \in \{1, \ldots, n\}$, define

$$\overrightarrow{t}^{(n)}_{\text{right},k} := x^{(n)}_k := a + k\,\delta_n \in [x^{(n)}_{k-1}, x^{(n)}_k] \ .$$

Then, from above, we see that

$$\begin{aligned} I = \int_a^b f(x)\,dx \ &= \ \lim_{n \longrightarrow \infty} S(f, \overrightarrow{P}^{(n)}, \overrightarrow{t}^{(n)}_{\text{right}}) \\ &= \ \lim_{n \longrightarrow \infty} \sum_{k=1}^{n} f(\overrightarrow{t}^{(n)}_{\text{right},k}) \left(x^{(n)}_k - x^{(n)}_{k-1}\right) \\ &= \ \lim_{n \longrightarrow \infty} \sum_{k=1}^{n} f(a + k\,\delta_n)\,\delta_n \ . \end{aligned}$$

17. Math 0235 : Lecture 17 : Fri 3/October/08

"Calculating integrals..."

Example[1] Fix arbitrary $a, b \in \mathbb{R}$ with $a < b$. Consider the *piecewise monotone* **function $f : [a, b] \longrightarrow \mathbb{R}$ given by**

$$f(x) := x^2 \ , \ \textbf{for all } x \in [a, b] \ .$$

Recall these facts from class and a recent Homework Assignment:

$$\sum_{k=1}^{n} k = \frac{1}{2} n\,(n+1) \quad \textbf{and} \quad \sum_{k=1}^{n} k^2 = \frac{1}{6} n\,(n+1)\,(2\,n+1)\ ,$$

for all $n \in \mathbb{N}$**. Also, recall an** *algebraic fact* **(that is easy to check):**

$$(b-a)\ \left(b^2 + a\,b + a^2\right) = b^3 - a^3\ .$$

From above,

$$\int_a^b x^2\,dx\ =\ \int_a^b f(x)\,dx = \lim_{n \longrightarrow \infty} \sum_{k=1}^{n} f(a + k\,\delta_n)\,\delta_n = \lim_{n \longrightarrow \infty} \sum_{k=1}^{n} (a + k\,\delta_n)^2\,\delta_n\ ,$$

where each

$$\delta_n := \frac{b-a}{n}\ .$$

Fix an arbitrary $n \in \mathbb{N}$**. Consider**

$$S_n\ :=\ \sum_{k=1}^{n} (a + k\,\delta_n)^2\,\delta_n = \delta_n \sum_{k=1}^{n} (a + k\,\delta_n)^2 = \delta_n \sum_{k=1}^{n} \left(a^2 + 2\,a\,k\,\delta_n + (k\,\delta_n)^2\right)\ .$$

Thus,

$$
\begin{aligned}
S_n &= \delta_n \sum_{k=1}^{n} \left(a^2 + 2\,a\,k\,\delta_n + (k\,\delta_n)^2\right) \\
&= \delta_n \left(\sum_{k=1}^{n} a^2 + \sum_{k=1}^{n} 2\,a\,k\,\delta_n + \sum_{k=1}^{n} k^2\,\delta_n^2\right) \\
&= \delta_n \left(n\,a^2 + 2\,a\,\delta_n \sum_{k=1}^{n} k + \delta_n^2 \sum_{k=1}^{n} k^2\right) \\
&= \delta_n \left(n\,a^2 + 2\,a\,\delta_n\, \frac{1}{2}\, n\,(n+1) + \delta_n^2\, \frac{1}{6}\, n\,(n+1)\,(2\,n+1)\right) \\
&= \delta_n\, n \left(a^2 + 2\,a\,\delta_n\, \frac{1}{2}\,(n+1) + \delta_n^2\, \frac{1}{6}\,(n+1)\,(2\,n+1)\right) \\
&= \delta_n\, n \left(a^2 + 2\,a\, \frac{1}{2}\,(\delta_n\,n + \delta_n) + \frac{1}{6}\,(\delta_n\,n + \delta_n)\,(2\,\delta_n\,n + \delta_n)\right) \quad .
\end{aligned}
$$

Now,

$$\delta_n \, n = \left(\frac{b-a}{n} \right) n = b - a \ .$$

Hence,

$$S_n = (b-a) \, \left(a^2 + a \left((b-a) + \delta_n \right) + \ \frac{1}{6} \left((b-a) + \delta_n \right) \left(2 \, (b-a) + \delta_n \right) \right) \ .$$

But

$$\lim_{n \longrightarrow \infty} \delta_n = \lim_{n \longrightarrow \infty} \frac{b-a}{n} = 0 \ .$$

From the *Algebra of Limits*, **we see that**

$$\begin{aligned} \int_a^b x^2 \, dx \ &= \ \lim_{n \longrightarrow \infty} S_n \\ &= \ (b-a) \, \left(a^2 + a \left((b-a) + 0 \right) + \ \frac{1}{6} \left((b-a) + 0 \right) \left(2 \, (b-a) + 0 \right) \right) \\ &= \ (b-a) \, \left(a^2 + a \, (b-a) + \ \frac{1}{6} \, (b-a) \, 2 \, (b-a) \right) \ . \end{aligned}$$

Therefore,

$$
\begin{aligned}
\int_a^b x^2\,dx &= \lim_{n\longrightarrow\infty} S_n \\
&= (b-a)\,\left(a^2 + a\,b - a^2 + \ \frac{1}{3}\,(b-a)^2\right) \\
&= \frac{1}{3}\,(b-a)\,\left(3\,a\,b + (b-a)^2\right) \\
&= \frac{1}{3}\,(b-a)\,\left(3\,a\,b + b^2 - 2\,a\,b + a^2\right) \\
&= \frac{1}{3}\,(b-a)\,\left(b^2 + a\,b + a^2\right) \\
&= \frac{1}{3}\,(b^3 - a^3)\ .
\end{aligned}
$$

The above example is the last lecture material that is **directly eligible** *for Exam 1...* Of course, there may be material in the rest of this lecture (Lecture 17) and in Lecture 18 that will help illuminate the Exam 1 material...

"Calculating integrals, continued..."

Example[2] Fix arbitrary $b \in \mathbb{R}$ **with** $0 < b$**. Consider the** *piecewise monotone* **function** $f : [0, b] \longrightarrow \mathbb{R}$ **given by**

$$f(x) := \cos(x) \ , \ \textbf{for all } x \in [0, b] \ .$$

In this example, the constant $a := 0$**.**

Let us *calculate* $I := \int_0^b \cos(x)\, dx \in \mathbb{R}$**. Firstly, we recall** *Euler's formulas for* cos **and** sin**.**

$$((1)) \qquad \cos(u) = \frac{e^{i\,u} + e^{-i\,u}}{2} \quad \textbf{and} \quad \sin(u) = \frac{e^{i\,u} - e^{-i\,u}}{2\,i} \ ,$$

for all $u \in \mathbb{R}$ **(or** $\mathbb{C}$**). Note that these two equations imply that**

$$((2)) \qquad e^{i\,u} = \cos(u) + i\,\sin(u) \quad \textbf{and} \quad e^{-i\,u} = \cos(u) - i\,\sin(u) \ ,$$

for all $u \in \mathbb{R}$ **(or** $\mathbb{C}$**).**

Also, by the *Addition Formula for* exp**: for all** $u \in \mathbb{C}$**, for every** $k \in \mathbb{N}$**, we have that**

$$\begin{aligned} ((3)) \qquad e^{k\,u} &= e^{[\ (1)^1 u \ + \ \dots \ + \ (1)^k u \ ]} \\ &= e^{(1)^1 u} \cdot \ \ \dots \ \ \cdot e^{(1)^k u} \\ &= (e^u)^k \ . \end{aligned}$$

Further, recall that for all $w \in \mathbb{C}$ with $w \neq 1$, for all $n \in \mathbb{N}$,

$$((4)) \qquad \sum_{k=1}^{n} w^k = w \sum_{k=1}^{n} w^{k-1} = w \sum_{j=0}^{n-1} w^j$$

[**using** : **the change of variables** $j = k - 1 \iff k = j + 1$]

$$= w \left(\frac{1 - w^{n-1+1}}{1 - w} \right) = w \left(\frac{1 - w^n}{1 - w} \right) .$$

From Lecture 16,

$$I := \int_0^b \cos(x)\, dx = \lim_{n \longrightarrow \infty} \sum_{k=1}^{n} \cos(0 + k\,\delta_n)\,\delta_n = \lim_{n \longrightarrow \infty} S_n ,$$

where, for all $n \in \mathbb{N}$,

$$\delta_n := \frac{b-0}{n} = \frac{b}{n} \quad \textbf{and} \quad S_n := \sum_{k=1}^{n} \cos(0 + k\,\delta_n)\,\delta_n .$$

Fix an arbitrary $n \in \mathbb{N}$. By $((1))$ **above,**

$$
\begin{aligned}
S_n \;:=\; & \sum_{k=1}^{n} \cos(0 + k\,\delta_n)\,\delta_n = \delta_n \sum_{k=1}^{n} \cos(k\,\delta_n) \\
=\; & \delta_n \sum_{k=1}^{n} \frac{e^{i\,k\,\delta_n} + e^{-i\,k\,\delta_n}}{2} = \delta_n \sum_{k=1}^{n} \frac{1}{2}\,\left(e^{i\,k\,\delta_n} + e^{-i\,k\,\delta_n}\right) \\
=\; & \delta_n \frac{1}{2} \sum_{k=1}^{n} \left(e^{i\,k\,\delta_n} + e^{-i\,k\,\delta_n}\right) = \frac{\delta_n}{2} \left(\sum_{k=1}^{n} e^{i\,k\,\delta_n} + \sum_{k=1}^{n} e^{-i\,k\,\delta_n} \right) \;.
\end{aligned}
$$

From $((3))$ **above, we see that**

$$
S_n \;=\; \frac{\delta_n}{2} \left(\sum_{k=1}^{n} \left(e^{i\,\delta_n}\right)^k + \sum_{k=1}^{n} \left(e^{-i\,\delta_n}\right)^k \right) \;.
$$

18. Math 0235 : Lecture 18 : Mon 6/October/08

"Calculating integrals..."

Example[2], continued...

Recall that $0 < \delta_n = b/n \longrightarrow 0$ **as** $n \longrightarrow \infty$**. So, there exists** $m \in \mathbb{N}$ **so that for all integer** $n \geq m$**,**

$$0 < \delta_n < \frac{\pi}{2} \ .$$

From our knowledge of *trigonometry* **it follows that for all** $n \geq m$**,**

$$0 < \cos(\delta_n) < 1 \quad \textbf{and} \quad 0 < \sin(\delta_n) < 1 \ .$$

Therefore, by $((2))$ **above, for all** $n \geq m$**,**

$$e^{i\,\delta_n} = \cos(\delta_n) + i\,\sin\delta_n \neq 1 + i\,0 = 1 \ ;$$

i.e., $e^{i\,\delta_n} \neq 1$**. Similarly, for all** $n \geq m$**,**

$$e^{-i\,\delta_n} = \cos(\delta_n) - i\,\sin\delta_n \neq 1 \ .$$

Note that

$$((5)) \qquad e^{i\,\delta_n}\,e^{-i\,\delta_n} = e^{i\,\delta_n + (-i\,\delta_n)} = e^0 = 1 \ .$$

From $((4))$**, with** $w = e^{i\,\delta_n}$ **and then** $w = e^{-i\,\delta_n}$**, we have that**

$$
\begin{aligned}
S_n &= \frac{\delta_n}{2}\left(\sum_{k=1}^{n}\left(e^{i\,\delta_n}\right)^k + \sum_{k=1}^{n}\left(e^{-i\,\delta_n}\right)^k\right)\\
&= \frac{\delta_n}{2}\left(e^{i\,\delta_n}\left(\frac{1-\left(e^{i\,\delta_n}\right)^n}{1-e^{i\,\delta_n}}\right) + e^{-i\,\delta_n}\left(\frac{1-\left(e^{-i\,\delta_n}\right)^n}{1-e^{-i\,\delta_n}}\right)\right)\\
&= \frac{\delta_n}{2}\left(\frac{1}{e^{-i\,\delta_n}}\left(\frac{1-e^{i\,\delta_n\,n}}{1-e^{i\,\delta_n}}\right) + \frac{1}{e^{i\,\delta_n}}\left(\frac{1-e^{-i\,\delta_n\,n}}{1-e^{-i\,\delta_n}}\right)\right) .
\end{aligned}
$$

Further note that $\delta_n \, n = (b/n) \, n = b$**. Thus, by** $((5))$**,**

$$
\begin{aligned}
S_n &= \frac{\delta_n}{2} \left(\left(\frac{1 - e^{i\,b}}{e^{-i\,\delta_n} - e^{-i\,\delta_n} \, e^{i\,\delta_n}} \right) + \left(\frac{1 - e^{-i\,b}}{e^{i\,\delta_n} - e^{i\,\delta_n} \, e^{-i\,\delta_n}} \right) \right) \\
&= \frac{\delta_n}{2} \left(\frac{1 - e^{i\,b}}{e^{-i\,\delta_n} - 1} + \frac{1 - e^{-i\,b}}{e^{i\,\delta_n} - 1} \right) \\
&= \frac{\delta_n}{2} \, \frac{\left(\frac{1}{\delta_n}\right)}{\left(\frac{1}{\delta_n}\right)} \left(\frac{1 - e^{i\,b}}{e^{-i\,\delta_n} - 1} + \frac{1 - e^{-i\,b}}{e^{i\,\delta_n} - 1} \right) \\
&= \frac{1}{2} \left(\frac{1 - e^{i\,b}}{\left(\frac{e^{-i\,\delta_n} - 1}{\delta_n} \right)} + \frac{1 - e^{-i\,b}}{\left(\frac{e^{i\,\delta_n} - 1}{\delta_n} \right)} \right) \\
&= \frac{1}{2} \left(\frac{1 - e^{i\,b}}{-i \, \left(\frac{e^{-i\,\delta_n} - 1}{-i\,\delta_n} \right)} + \frac{1 - e^{-i\,b}}{i \, \left(\frac{e^{i\,\delta_n} - 1}{i\,\delta_n} \right)} \right) \quad .
\end{aligned}
$$

Now, we know that $0 < \delta_n := b/n \longrightarrow 0$ **as** $n \longrightarrow \infty$**. Thus, by** *the Algebra of Limits***,**

$$\lim_{n \longrightarrow \infty} -i\,\delta_n = 0 \quad \textbf{and} \quad \lim_{n \longrightarrow \infty} i\,\delta_n = 0 \ .$$

Also, each $-i\,\delta_n \neq 0$ **and each** $i\,\delta_n \neq 0$**. So, by earlier work, we know that**

$$\lim_{n \longrightarrow \infty} \frac{e^{-i\,\delta_n} - 1}{-i\,\delta_n} = 1 \quad \textbf{and} \quad \lim_{n \longrightarrow \infty} \frac{e^{i\,\delta_n} - 1}{i\,\delta_n} = 1 \ .$$

Hence, by *the Algebra of Limits* **and** $((1))$ **above,**

$$\begin{aligned}
\int_0^b \cos(x)\,dx &= \lim_{n \longrightarrow \infty} S_n \\
&= \frac{1}{2}\left(\frac{1 - e^{i\,b}}{-i\ (1)} + \frac{1 - e^{-i\,b}}{i\ (1)}\right) = \frac{1}{2\,i}\left(\frac{1 - e^{i\,b}}{-1} + \frac{1 - e^{-i\,b}}{1}\right) \\
&= \frac{1}{2\,i}\left(-(1 - e^{i\,b}) + 1 - e^{-i\,b}\right) = \frac{1}{2\,i}\left(-1 + e^{i\,b} + 1 - e^{-i\,b}\right) \\
&= \frac{1}{2\,i}\left(e^{i\,b} - e^{-i\,b}\right) = \sin(b) \ .
\end{aligned}$$

19. Math 0235 : Lecture 19 : Wed 8/October/08

"Calculating integrals..."

Example[3].

Fix arbitrary $b \in \mathbb{R}$ **with** $0 < b$**. Consider the** *piecewise monotone* **function** $f : [0, b] \longrightarrow \mathbb{R}$ **given by**

$$f(x) := x^4 \ , \ \textbf{for all } x \in [0, b] \ .$$

In this example, the constant $a := 0$**.**

Firstly, we recall the following *"funny powers" formula***.**

$$((1)) \qquad \sum_{k=1}^{n} k^{\langle 4 \rangle} = \frac{n^{\langle 5 \rangle}}{5} \ , \ \textbf{for all } n \in \mathbb{N} \ .$$

From Lecture 16,

$$I := \int_0^b x^4 \, dx \ = \ \lim_{n \longrightarrow \infty} \sum_{k=1}^{n} (0 + k\,\delta_n)^4 \, \delta_n = \lim_{n \longrightarrow \infty} S_n \ ,$$

where, for all $n \in \mathbb{N}$**,**

$$\delta_n := \frac{b - 0}{n} = \frac{b}{n} \ \ \textbf{and} \ \ S_n := \sum_{k=1}^{n} (0 + k\,\delta_n)^4 \, \delta_n \ .$$

Fix an arbitrary $n \in \mathbb{N}$. By $((1))$ above,

$$\begin{aligned} S_n &:= \sum_{k=1}^{n} (0 + k\,\delta_n)^4\,\delta_n = \delta_n \sum_{k=1}^{n} (k\,\delta_n)^4 \\ &= \delta_n \sum_{k=1}^{n} k^4\,\delta_n^4 = \delta_n\,\delta_n^4 \sum_{k=1}^{n} k^4 \\ &= \delta_n^5 \sum_{k=1}^{n} k^4 \ . \end{aligned}$$

Note that, for each $k \in \{1, \ldots, n\}$,

$$k^4 \le k\,(k+1)\,(k+2)\,(k+3) = k^{\langle 4 \rangle} \ .$$

Let

$$U_n := \delta_n^5 \sum_{k=1}^{n} k^{\langle 4 \rangle} \ .$$

Since $\delta_n = b/n > 0$**, it follows that**

$$S_n = \delta_n^5 \sum_{k=1}^{n} k^4 \le \delta_n^5 \sum_{k=1}^{n} k^{\langle 4 \rangle} = U_n \ .$$

Further, from $((1))$ **above,**

$$\begin{aligned} U_n &= \delta_n^5 \sum_{k=1}^{n} k^{\langle 4 \rangle} = \delta_n^5 \, \frac{n^{\langle 5 \rangle}}{5} \\ &= \left(\frac{b}{n}\right)^5 \frac{n\,(n+1)\,(n+2)\,(n+3)\,(n+4)}{5} \\ &= \frac{b^5}{5} \left(\frac{n}{n}\right) \left(\frac{n+1}{n}\right) \left(\frac{n+2}{n}\right) \left(\frac{n+3}{n}\right) \left(\frac{n+4}{n}\right) \\ &= \frac{b^5}{5}\,(1) \left(1+\frac{1}{n}\right) \left(1+\frac{2}{n}\right) \left(1+\frac{3}{n}\right) \left(1+\frac{4}{n}\right) \ . \end{aligned}$$

Since $\lim_{n \longrightarrow \infty} 1/n = 0$**, we see by the** *Algebra of Limits* **that**

$$\lim_{n \longrightarrow \infty} U_n = \frac{b^5}{5}\,(1)\,(1+0)\,(1+0)\,(1+0)\,(1+0) = \frac{b^5}{5} \ .$$

Next, fix $n \in \mathbb{N}$ with $n \geq 4$. Note that, for each $k \in \{4, \ldots, n\}$,

$$k^4 \geq (k-3)\,(k-2)\,(k-1)\,k = (k-3)^{\langle 4 \rangle} \ .$$

Let

$$Q_n := \delta_n^5 \sum_{k=4}^{n} (k-3)^{\langle 4 \rangle} \ .$$

Since $\delta_n = b/n > 0$ and each $k > 0$, it follows that

$$S_n = \delta_n^5 \sum_{k=1}^{n} k^4 \geq \delta_n^5 \sum_{k=4}^{n} k^4 \geq \delta_n^5 \sum_{k=4}^{n} (k-3)^{\langle 4 \rangle} = Q_n \ .$$

Further, from ((1)) **above, using the** *change of variables* $[j = k - 3 \iff k = j + 3]$, **we see that**

$$
\begin{aligned}
Q_n &= \delta_n^5 \sum_{k=4}^{n} (k-3)^{\langle 4 \rangle} = \delta_n^5 \sum_{j=1}^{n-3} j^{\langle 4 \rangle} = \delta_n^5 \, \frac{(n-3)^{\langle 5 \rangle}}{5} \\
&= \left(\frac{b}{n}\right)^5 \frac{(n-3)(n-2)(n-1)\,n\,(n+1)}{5} \\
&= \frac{b^5}{5} \left(\frac{n-3}{n}\right) \left(\frac{n-2}{n}\right) \left(\frac{n-1}{n}\right) \left(\frac{n}{n}\right) \left(\frac{n+1}{n}\right) \\
&= \frac{b^5}{5} \left(1 - \frac{3}{n}\right) \left(1 - \frac{2}{n}\right) \left(1 - \frac{1}{n}\right) (1) \left(1 + \frac{1}{n}\right) .
\end{aligned}
$$

Since $\lim_{n \longrightarrow \infty} 1/n = 0$, **we see by the** *Algebra of Limits* **that**

$$\lim_{n \longrightarrow \infty} Q_n = \frac{b^5}{5} \, (1-0) \, (1-0) \, (1-0) \, (1) \, (1+0) = \frac{b^5}{5} \; .$$

Consequently, by *the Squeeze Theorem*, **since** $[Q_n \leq S_n \leq U_n, \forall n \geq 4]$,

$$\int_0^b x^4 \, dx = \lim_{n \longrightarrow \infty} S_n = \frac{b^5}{5} \; .$$

HA.L19 [1].

In a manner similar to the calculation of $\int_0^b \cos(x)\,dx$ **and** $\int_0^b x^4\,dx$ **(where** $b > 0$**) in Lectures 17, 18 and 19,** *calculate*

(1) $\int_0^b \sin(x)\,dx$**, for all** $b > 0$**;**

(2) $\int_0^b x^5\,dx$**, for all** $b > 0$**; and**

(3) $\int_0^b x^\nu\,dx$**, for all** $b > 0$ **and for all** $\nu \in \mathbb{N}$**.**

20. Math 0235 : Lecture 20 : Fri 10/October/08

HA.L20.

[**1**] **Using ideas from Lecture 20 (and earlier),** *calculate* $\int_a^b \sin(x)\,dx$**, for all** $a, b \in \mathbb{R}$**.**

[**2**] *Read and carefully note:* **Stewart, Section 9.3 "The Dot Product", pages 651-655.**

[**3**] **Do questions 4, 5, 7, 11, 16, [17 (b) and (d)], 20, 27, 31, 35 and 39: on pages 656-657 of Stewart.**

[4] Do Question 43, p. 657 of Stewart: using only the *algebraic "Properties of the Dot Product", page 654 of Stewart".*

Hint (1): Note that $\overrightarrow{v} \bullet \overrightarrow{v} \geq 0$**, for all vectors** $\overrightarrow{v}$**.**

Hint (2): Expand $(\overrightarrow{a} - \lambda \overrightarrow{b}) \bullet (\overrightarrow{a} - \lambda \overrightarrow{b})$ **and choose** λ *appropriately*...

[5] Do questions 44 and 45: on page 657 of Stewart.

Definition 20.1. Fix arbitrary $a, b \in \mathbb{R}$ with $b < a$. Suppose that the function $f : [b, a] \longrightarrow \mathbb{R}$ is *Riemann-integrable* on $[b, a]$. We define *"the integral of f from a to b"*, $\int_a^b f(x)\,dx$, by

$$\int_a^b f(x)\,dx := -\int_b^a f(x)\,dx \ .$$

We also define

$$\int_a^a f(x)\,dx := 0 \ .$$

Note: it follows that *for all real numbers* $a, b \in \mathbb{R}$**,**

$$(\ddagger) \qquad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \ .$$

[A Useful Fact]. **The formula**

$$(\#) \qquad \int_a^b f(x)\,dx = \lim_{n \longrightarrow \infty} \sum_{k=1}^{n} f(a + k\,\delta_n)\,\delta_n \ ,$$

where

$$\delta_n := \frac{b-a}{n} \ ,$$

which is true whenever $a < b$**,** *remains true* **when** $b < a$ **and when** $b = a$**.**

Another useful fact is:

[¡WOW!] Proposition. Fix arbitrary $u, v \in \mathbb{R}$ with $u < v$. Let $f : [u, v] \longrightarrow \mathbb{R}$ be a Riemann-integrable function. Let $a, b, c \in [u, v]$. Then each of the integrals below exists in $\mathbb{R}$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \ .$$

"Applications of Proposition **[¡WOW!]** *and* **formula** (#)**".**

Similarly to HA.L19 [1] (3) (where $b > 0$), we can show from formula $(\#)$ **that for all** $b \leq 0$**,**

$$(\dagger) \qquad \int_0^b x^\nu \, dx = \frac{b^{\nu+1}}{\nu+1} \ , \ \textbf{for all } \nu \in \mathbb{N} \ .$$

In summary, formula $(\dagger)$ **is true** *for all real numbers* b**.**

Consequently, for all $a, b \in \mathbb{R}$**, for all** $\nu \in \mathbb{N}$**, using** $(\ddagger)$**,** $(\dagger)$ **and [¡WOW!] with** $c := 0$**, we see that**

$$\begin{aligned}
\int_a^b x^\nu \, dx &= \int_a^0 x^\nu \, dx + \int_0^b x^\nu \, dx \\
&= -\int_0^a x^\nu \, dx + \int_0^b x^\nu \, dx \\
&= -\frac{a^{\nu+1}}{\nu+1} + \frac{b^{\nu+1}}{\nu+1} \\
&= \frac{b^{\nu+1} - a^{\nu+1}}{\nu+1} \ .
\end{aligned}$$

Next, recall that we calculated that

$$\int_0^b \cos(x)\,dx = \sin(b) \ ,$$

for all $b > 0$**. Using formula** $(\#)$ **when** $b \leq 0$**, a similar calculation reveals that**

$$\int_0^b \cos(x)\,dx = \sin(b) \ ,$$

for all $b \leq 0$**. In summary, this formula is true for all** $b \in \mathbb{R}$**.**

Consequently, using $(\ddagger)$ **and [¡WOW!], we see that for all** $a, b \in \mathbb{R}$**,**

$$\begin{aligned} \int_a^b \cos(x)\,dx &= \int_a^0 \cos(x)\,dx + \int_0^b \cos(x)\,dx \\ &= -\int_0^a \cos(x)\,dx + \int_0^b \cos(x)\,dx \\ &= -\sin(a) + \sin(b) \\ &= \sin(b) - \sin(a) \ . \end{aligned}$$

The above results are special cases of "The Fundamental Theorem of Calculus"... Let's now introduce the concept of the *"the derivative...*

Definition 20.2 (The Sequential Definition of Derivative)**.** Let J be an *interval* and $c \in J$. Let $f : J \longrightarrow \mathbb{R}$ be a function. We say that f *is differentiable at* c, if there exists $L \in \mathbb{R}$ such that for every sequence $(h_n)_{n \in \mathbb{N}}$ with

$$\lim_{n \longrightarrow \infty} h_n = 0 \quad , \quad [h_n \neq 0 \ , \ \text{for all } n \in \mathbb{N}] \ ,$$

and each $c + h_n \in J$, it follows that

$$\lim_{n \longrightarrow \infty} \frac{f(c + h_n) - f(c)}{h_n} = L \ .$$

We denote this *limit* L by $f'(c)$, and call it *the derivative of* f *at* c.

21. Math 0235 : Lecture 21 : Tues 14/October/08

"Calculating derivatives..."

Definition 21.1 (The Sequential Definition of Derivative)**.** [Note: This definition is equivalent to the usual one...] Let J be an *interval* and $c \in J$. Let $f : J \longrightarrow \mathbb{R}$ be a function. We say that f is *differentiable* or *locally linear* at c, if there exists $L \in \mathbb{R}$ such that for every sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with

$$\lim_{n\longrightarrow\infty} h_n = 0 \;\;,\;\; [h_n \neq 0\;,\; \text{for all } n \in \mathbb{N}]\;,$$

and each $c + h_n \in J$, it follows that

$$\lim_{n\longrightarrow\infty} \frac{f(c+h_n) - f(c)}{h_n} = L\;.$$

We denote this *common limit* L by $f'(c)$, and call it *the derivative of f at c.*

Note that

$$Q_n := \frac{f(c+h_n) - f(c)}{h_n}$$

is the *slope of the chord C* **on graph**(f) **joining the points** $(c, f(c))$ **and** $(c+h_n, f(c+h_n))$**. Extend this chord to a line** ℓ **with slope** Q_n**. As** $n \longrightarrow \infty$**, this line** *approaches* **the** *line T with slope $f'(c)$ through* *the point* $(c, f(c)$. **We** *define* **this line to be** *the tangent to graph*(f) *at* $(c, f(c))$. **Note that**

$$T := \{(x, y) \in \mathbb{R}^2 : y - f(c) = f'(c)\,(x - c)\}\ .$$

[In class, we drew a "typical graph of f**", to illustrate the above ideas...]**

Also,

$$Q_n := \frac{f(c + h_n) - f(c)}{h_n}$$

is the *average rate of change* **of** f **over the interval** $[c, c+h_n]$**, if** $h_n > 0$**; or** $[c + h_n, c]$**, if** $h_n < 0$**.**

In the above definition, we require that for all *null sequences* $(h_n)_{n\in\mathbb{N}}$**, as described above,** Q_n **converges to** *the same limit* L**. Otherwise it is easy to draw the graph of functions** f **for which (for example), for all such null sequences with each** $h_n > 0$**,** Q_n **tends to a certain number** L_1**; while for all such null sequences with each** $h_n < 0$**,** Q_n **tends to a another number** L_2**. Such functions** f **are** *not differentiable at* c**.**

[In class, we drew a "typical graph of f**", to illustrate the above ideas...]**

Example [1]. Let $J := \mathbb{R}$ **and the function** $f : J \longrightarrow \mathbb{R}$ **be defined by**

$$f(x) := x^3 \ , \ \textbf{for all } x \in J = \mathbb{R} \ .$$

Fix an arbitrary point $c \in J = \mathbb{R}$**. Next, fix an arbitrary sequence** $(h_n)_{n\in\mathbb{N}}$ **in** $\mathbb{R}$ **with**

$$\lim_{n \longrightarrow \infty} h_n = 0 \ \ \textbf{and} \ \ [h_n \neq 0 \ , \ \textbf{for all } n \in \mathbb{N}] \ .$$

(Note that each $c + h_n \in J = \mathbb{R}$**.) Further, fix an arbitrary** $n \in \mathbb{N}$ **and consider**

$$Q_n := \frac{f(c+h_n) - f(c)}{h_n} \ .$$

Using *Pascal's triangle* **to help us** *expand* $(c+h_n)^3$**, we see that**

$$\begin{aligned} Q_n \ &= \ \frac{f(c+h_n) - f(c)}{h_n} = \frac{1}{h_n} \ \left((c+h_n)^3 - c^3\right) \\ &= \ \frac{1}{h_n} \ \left(c^3 + 3\,c^2\,h_n + 3\,c\,h_n^2 + h_n^3 - c^3\right) \ . \end{aligned}$$

Thus, taking out a *common factor* **of** h_n**, we have**

$$\begin{aligned} Q_n &= \frac{1}{h_n} \left(3\,c^2\,h_n + 3\,c\,h_n^2 + h_n^3\right) \\ &= \frac{1}{h_n}\,h_n \left(3\,c^2 + 3\,c\,h_n + h_n^2\right) \\ &= 3\,c^2 + 3\,c\,h_n + h_n^2 \ . \end{aligned}$$

Now,

$$\lim_{n \longrightarrow \infty} h_n = 0 \ .$$

Consequently, by the *Algebra of Limits (A.o.L.)*,

$$\lim_{n \longrightarrow \infty} Q_n = \lim_{n \longrightarrow \infty} \left(3\,c^2 + 3\,c\,h_n + h_n^2\right) = 3\,c^2 + 3\,c\,(0) + (0)^2 = 3\,c^2 \ .$$

In summary, we have shown that for all $c \in \mathbb{R}$**,**

$$f'(c) = 3\,c^2 \ .$$

Example [2]. Let $J := \mathbb{R}$**,** $\nu \in \mathbb{N}$ **and the function** $f : J \longrightarrow \mathbb{R}$ **be defined by**

$$f(x) := x^\nu \ , \ \textbf{for all } x \in J = \mathbb{R} \ .$$

By similar arguments to above, using *the Binomial Theorem*, **we can show from** *the definition of the derivative* **that for all** $c \in \mathbb{R}$,

$$f'(c) = \nu\, c^{\nu-1} \ .$$

Example [3]. Let $J := \mathbb{R}$ **and the function** $f : J \longrightarrow \mathbb{R}$ **be defined by**

$$f(x) := e^x = \exp(x) \ , \ \textbf{for all } x \in J = \mathbb{R} \ .$$

Fix an arbitrary point $c \in J = \mathbb{R}$. **Next, fix an arbitrary sequence** $(h_n)_{n\in\mathbb{N}}$ **in** $\mathbb{R}$ **with**

$$\lim_{n\longrightarrow\infty} h_n = 0 \ \ \textbf{and} \ \ [h_n \neq 0 \ , \ \textbf{for all } n \in \mathbb{N}] \ .$$

(Note that each $c + h_n \in J = \mathbb{R}$.**) Further, fix an arbitrary** $n \in \mathbb{N}$ **and consider**

$$Q_n := \frac{f(c+h_n) - f(c)}{h_n} \ .$$

Using *the addition formula for* exp **to help us** *expand* $\exp(c+h_n)$**, we see that**

$$\begin{aligned} Q_n &= \frac{f(c+h_n)-f(c)}{h_n} = \frac{1}{h_n}\left(e^{c+h_n}-e^c\right) \\ &= \frac{1}{h_n}\left(e^c\,e^{h_n}-e^c\right) \ . \end{aligned}$$

Thus, taking out a *common factor* **of** e^c**, we have**

$$\begin{aligned} Q_n &= \frac{1}{h_n}\,e^c\left(e^{h_n}-1\right) \\ &= e^c\left(\frac{e^{h_n}-1}{h_n}\right) \ . \end{aligned}$$

Now,

$$\lim_{n\longrightarrow\infty} h_n = 0 \ \ \textbf{and} \ \ \left[h_n \neq 0 \ , \ \textbf{for all } n \in \mathbb{N}\right] \ .$$

By earlier work, using the *A.o.L. Super Fact* **and the** *power series definition* **of** exp**, we know it follows that**

$$\lim_{n\longrightarrow\infty}\left(\frac{e^{h_n}-1}{h_n}\right) = 1 \ .$$

Consequently, by the *Algebra of Limits (A.o.L.)*,

$$\lim_{n\longrightarrow\infty} Q_n = \lim_{n\longrightarrow\infty} e^c \left(\frac{e^{h_n}-1}{h_n}\right) = e^c\,(1) = e^c \ .$$

In summary, we have shown that for all $c \in \mathbb{R}$**,**

$$f'(c) = e^c \ .$$

Example [4]. Let $J := \mathbb{R}$ **and the function** $f : J \longrightarrow \mathbb{R}$ **be defined by**

$$f(x) := \sin(x) \ , \ \textbf{for all } x \in J = \mathbb{R} \ .$$

Fix an arbitrary point $c \in J = \mathbb{R}$**. Next, fix an arbitrary sequence** $(h_n)_{n\in\mathbb{N}}$ **in** $\mathbb{R}$ **with**

$$\lim_{n\longrightarrow\infty} h_n = 0 \ \ \textbf{and} \ \ [h_n \neq 0 \ , \ \textbf{for all } n \in \mathbb{N}] \ .$$

(Note that each $c + h_n \in J = \mathbb{R}$**.) Further, fix an arbitrary** $n \in \mathbb{N}$ **and consider**

$$Q_n := \frac{f(c+h_n) - f(c)}{h_n} \ .$$

Using *the addition formula for* $\sin$ **to help us** *expand* $\sin(c+h_n)$**, we see that**

$$\begin{aligned} Q_n &= \frac{f(c+h_n)-f(c)}{h_n} = \frac{1}{h_n}\ (\sin(c+h_n)-\sin(c)) \\ &= \frac{1}{h_n}\ (\sin(c)\ \cos(h_n)+\cos(c)\ \sin(h_n)-\sin(c)) \\ &= \frac{1}{h_n}\big(\sin(c)\,(\cos(h_n)-1)+\cos(c)\ \sin(h_n)\big) \\ &= \sin(c)\ \left(\frac{\cos(h_n)-1}{h_n}\right)+\cos(c)\ \left(\frac{\sin(h_n)}{h_n}\right)\ . \end{aligned}$$

Now,

$$\lim_{n\longrightarrow\infty} h_n = 0 \ \ \textbf{and} \ \ [h_n \neq 0\ ,\ \textbf{for all } n\in\mathbb{N}]\ .$$

By earlier work, using the *A.o.L. Super Fact* **and the** *power series definitions* **of** $\sin$ **and** $\cos$**, we know it follows that**

$$\lim_{n\longrightarrow\infty}\left(\frac{\cos(h_n)-1}{h_n}\right)=0 \ \ \textbf{and} \ \ \lim_{n\longrightarrow\infty}\left(\frac{\sin h_n}{h_n}\right)=1\ .$$

Consequently, by the *Algebra of Limits (A.o.L.)*,

$$\lim_{n \longrightarrow \infty} Q_n = \lim_{n \longrightarrow \infty} \left[\sin(c) \left(\frac{\cos(h_n) - 1}{h_n} \right) + \cos(c) \left(\frac{\sin(h_n)}{h_n} \right) \right]$$
$$= \sin(c)\ (0) + \cos(c)\ (1) = \cos(c)\ .$$

In summary, we have shown that for all $c \in \mathbb{R}$**,**

$$f'(c) = \cos(c)\ .$$

HA.L21 [1] Using the *definition of the derivative*, **and in a similar manner to Example [4] above:** *calculate* $f'(c)$, **for all** $c \in \mathbb{R}$**; where**

$$(1) \qquad f(x) := \cos(x)\ , \text{ for all } x \in J := \mathbb{R}\ ;$$

and

$$(2) \qquad f(x) := \frac{1}{x}\ , \text{ for all } x \in J := (0, \infty)\ .$$

We next began to discuss *the natural logarithm function* $\ln : (0, \infty) \longrightarrow \mathbb{R}$**. We will re-cap and continue this discussion in Lecture 22...**

22. MATH 0235 : LECTURE 22 : WED 15/OCTOBER/08

"Calculating derivatives from the definition, continued..."

HA.L22 [1] Using the *definition of the derivative*, **and in a similar manner to the examples in Lecture 21:** *calculate* $f'(c)$, **for all** $c \in \mathbb{R}$**; where**

$$(1) \qquad f(x) := x^5 \ , \ \textbf{for all } x \in J := \mathbb{R} \ ;$$

$$(2) \qquad f(x) := x^{3/2} \ , \ \textbf{for all } x \in J := (0, \infty) \ ;$$

and

$$(3) \qquad f(x) := x^{1/5} \ , \ \textbf{for all } x \in J := (0, \infty) \ .$$

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